TOPOLOGICAL QUANTUM GROUPS AND HOPF ALGEBRAS

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Algebraic quantum groups and groupoids

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- Lecture 1: Finite quantum groups
- Lecture 2: Multiplier Hopf *-algebras
- Lecture 3: Algebraic quantum groups and duality
- Lecture 4: Towards operator algebraic quantum groups
- Lecture 5: Algebraic quantum groupoids

Lecture 1: Finite quantum groups and duality
(5^{th} September 2016)

- Introduction
- Finite-dimensional Hopf *-algebras
- The dual of a finite-dimensional Hopf *-algebra
- The Heisenberg commutation relations
- An example
- Further reflections and some conclusions
- References

1. Introduction (A few well-known historical facts)

The famous theorem of Pontryagin states that the dual of an abelian locally compact group G is again an abelian locally compact group $\widehat{G}$. Here, $\widehat{G}$ is the group of continuous homomorphisms from G to the group $\mathbb{T}$ of complex numbers with modulus 1, endowed with the topology of uniform convergence on compact sets. The theorem dates from 1939. Since then, various attempts have been made by various people to extend this result to the non-abelian case.

The first theory that restored the self-duality came with the Kac algebras in the late 60's, early 70's. With independent work by Kac and Vainerman and by Enock and Schwartz. This theory however was unsatisfactory, for different reasons:

- The set of axioms is quite complicated.
- The theory is difficult and requires many technical skills.
- But most importantly, the concept is too restrictive.

For a Kac algebra, the antipode S (corresponding to taking the inverse in the group case) is assumed to satisfy $S^2 = \iota$, the identity map. This assumption was quite natural at that time, but later, turned out to be too restrictive.

The new ideas came simultaneously with the theory of compact quantum groups (and the $SU_q(2)$ example) by Woronowicz (1987) and the work of Drinfel'd and Jimbo on quantum groups (1986). They gave examples where the square of the antipode is not the identity map. It is a bit strange that this case was not considered earlier as it was certainly known in Hopf algebra theory before.

These new examples triggered a new search for a concept, generalizing the Kac algebras, and still within a self-dual setting. This eventually led to work by Masuda, Nakagami and Woronowicz, on one side (1994), and by Kustermans and Vaes on the other side (1999). The notion as developed by Kustermans and Vaes is now widely considered as the correct one for a locally compact quantum group.Also the preceding work on multiplicative unitaries, by Baaj and Skandalis (1993) (and by Woronowicz and Soltan later) played an important role.

In any case, in the development of quantum groups in the operator algebraic setting, resulting in the theory of locally compact quantum groups, duality has always played a crucial role. This is less so in the purely algebraic theory.

The theory of locally compact quantum groups is a major achievement in the operator algebra approach to quantum groups. It involves aspects of a purely algebraic nature on the one hand and of a topological nature on the other hand. The interplay between the two is far from trivial.

If we want to understand the difficulties that arise here, it is important to have a sound knowledge of the purely algebraic aspects before passing to the more complicated topological theory. We feel that the best place to start this, is the finite-dimensional case. Therefore, we begin these lectures with a study of finite-dimensional Hopf algebras and their duality.

2. Hopf *-algebras

Let A be an associative algebra over the field $\mathbb{C}$ of complex numbers. Assume that it has an identity, denoted by 1. The following notions are crucial for the rest of the lectures.

A *coproduct* on A is a homomorphism $\Delta : A \to A \otimes A$ satisfying

$$(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta,$$

where ι denotes the identity map from A to itself.

A *counit* is a linear map $\varepsilon : A \to \mathbb{C}$ satisfying

$$(\varepsilon \otimes \iota)\Delta(a) = a \text{ and } (\iota \otimes \varepsilon)\Delta(a) = a,$$

for all $a \in A$. It is unique (if it exists).

An *antipode* S is a linear map from A to itself satisfying

$$m(S \otimes \iota)\Delta(a) = \varepsilon(a)1 \text{ and } m(\iota \otimes S)\Delta(a) = \varepsilon(a)1,$$

for all a in A. Here m denotes multiplication, seen as a linear map from $A \otimes A$ to A. Also an antipode is unique if it exists.

This takes us to the following important definition.

Definition 2.1 A *Hopf algebra* is a pair (A, Δ) of an algebra A and a coproduct Δ on A such that there exists a counit ε and an antipode S. If moreover A is a *-algebra and Δ a *-homomorphism, then the pair is called a Hopf *-algebra.

Proposition 2.1 *Let (A, Δ) be a Hopf algebra. The counit is a homomorphism. In the case of a Hopf *-algebra, it is a *-homomorphism. The antipode is an anti-homomorphism and in the case of a Hopf *-algebra it satisfies $S(S(a)^*)^* = a$ for all a. It also flips the coproduct in the sense that $\Delta(S(a)) = \zeta(S \otimes S)\Delta(a)$ for all a where ζ is the flip map $b \otimes c \mapsto c \otimes b$ on $A \otimes A$.*

Here are the two basic examples:

Proposition 2.2 *Let G be a finite group. Denote by $C(G)$ the algebra of complex functions on G with pointwise operations. We identity $C(G \times G)$ with $C(G) \otimes C(G)$. Define $\Delta : C(G) \to C(G) \otimes C(G)$ by $\Delta(f)(p, q) = f(pq)$. Then the pair $(C(G), \Delta)$ is a Hopf algebra. The counit is given by $\varepsilon(f) = f(e)$ where e is the identity in G. The antipode is given by $S(f)(p) = f(p^{-1})$ where p^{-1} is the inverse of the element p.*

It is a Hopf *-algebra for the obvious involution $f \mapsto \overline{f}$ where $\overline{f}(p) = \overline{f(p)}$.

Remark 2.1 This result is no longer true when G is infinite. We will need a more general notion than a Hopf algebra (cf. later).

Proposition 2.3 *Let G be a group (not necessarily finite). Consider the group algebra $\mathbb{C}G$ and denote by $p \mapsto \lambda_p$ the canonical embedding of G in $\mathbb{C}G$. Then $\Delta(\lambda_p) = \lambda_p \otimes \lambda_p$ defines a coproduct on $\mathbb{C}G$ making it into a Hopf algebra.*

It is a Hopf *-algebra if we define $\lambda_p^* = \lambda_{p^{-1}}$ for all p.

Remark 2.2 Unlike the previous example, this also works when G is infinite.

Remark 2.3 For the first example above, the algebra is abelian. For the second one, the coproduct is coabelian in the sense that $\Delta = \zeta\Delta$. We will see that these examples are 'dual' to each other.

But first, let us consider the following example with a non-abelian algebra and a coproduct that is not cocommutative.

Proposition 2.4 *Let λ be any non-zero complex number. Let A be the unital algebra generated by an invertible element a and an element b satisfying $ab = \lambda ba$. There is a coproduct Δ on A given by*

$$\Delta(a) = a \otimes a \text{ and } \Delta(b) = a \otimes b + b \otimes 1.$$

The counit is given by $\varepsilon(a) = 1$ *and* $\varepsilon(b) = 0$ *while the antipode is given by*

$$S(a) = a^{-1} \text{ and } S(b) = -a^{-1}b.$$

It is a Hopf *-algebra when λ has modulus 1 and if we let a and b be self-adjoint elements.

3. The dual Hopf algebra

Let (A, Δ) be a finite-dimensional Hopf algebra. Denote by B the linear dual space A' of A. We identity $B \otimes B$ with the dual $(A \otimes A)'$ of $A \otimes A$.

Proposition 3.1 *Define a product on* B *by* $(fg)(a) = (f \otimes g)(\Delta(a))$. *This makes* B *into an associative algebra. It is unital and the unit is given by* ε. *There exists a coproduct* Δ *on* B *given by* $\Delta(f)(a \otimes a') = f(aa')$. *The pair* (B, Δ) *is again a Hopf algebra. The counit on* B *is given by* $f \mapsto f(1)$ *where* 1 *is the identity in* A. *The antipode on* B *is given* $(S(f))(a) = f(S(a))$. *If* (A, Δ) *is a Hopf* **-algebra, then so is* (B, Δ) *for the involution on* B *defined by* $f^*(a) = \overline{f(S(a)^*)}$.

The two group examples are dual to each other. The duality is given by

$$\langle f, \lambda_p \rangle = f(p).$$

Definition 3.1 Let (A, Δ) and (B, Δ) be Hopf algebras. We call it a *dual pair* if there is a non-degenerate bilinear form $(a, b) \mapsto \langle a, b \rangle$ satisfying

$$\langle a, bb' \rangle = \langle \Delta(a), b \otimes b' \rangle \text{ and } \langle aa', b \rangle = \langle a \otimes a', \Delta(b) \rangle$$

for all $a, a' \in A$ and $b, b' \in B$.

Remark 3.1 We consider the tensor product pairing here as a map on the Cartesian product of $A \otimes A$ with $B \otimes B$. In the case of Hopf *-algebras, we require $\langle a, b^* \rangle = \langle S(a)^*, b \rangle^-$ for all a, b. If (A, Δ) is a Hopf (*-)algebra and (B, Δ) its dual, then we have a dual pair of Hopf (*-)algebras.

The following observation is important.

Proposition 3.2 *Assume that* (A, Δ) *and* (B, Δ) *are a dual pair of Hopf* **-algebras. Then the underlying Hopf* **-algebra structures of* A *and* B *are completely determined by the pairing of the* **-algebras* A *and* B.

- The coproducts are adjoint to the products.
- The counits are given by
$$\varepsilon(a) = \langle a, 1 \rangle \text{ and } \varepsilon(b) = \langle 1, b \rangle$$
for $a \in A$ and $b \in B$.
- The antipodes are given by the formulas
$$a, b^* \rangle = \langle S(a)^*, b \rangle^- \text{ and } \langle a^*, b \rangle = \langle a, S(b)^* \rangle^-$$
for $a \in A$ and $b \in B$.

4. Commutation relations

Consider a pairing of two finite-dimensional *-algebras A and B. It induces a left and a right action of B on A as follows:

Notation 4.1 Take $x \in A$ and $b \in B$. Define $b \triangleright x$ and $x \triangleleft b$ in A by

$$\langle b \triangleright x, b' \rangle = \langle x, b'b \rangle \text{ and } \langle x \triangleleft b, b' \rangle = \langle x, bb' \rangle.$$

Also A acts on A by left and right multiplication. We have the following commutation rules:

Proposition 4.1 *With the same notation as above, we get*

$$b \triangleright (ax) = \sum_{(a),(b)} \langle a_{(2)}, b_{(1)} \rangle a_{(1)} (b_{(2)} \triangleright x),$$

where $\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}$ *and* $\Delta(b) = \sum_{(b)} b_{(1)} \otimes b_{(2)}$.

Let (A, Δ_A) and (B, Δ_B) be as before. Define $W \in B \otimes A$ by $\langle W, a \otimes b \rangle = \langle a, b \rangle$, where we use the pairing between $B \otimes A$ and $A \otimes B$.

Proposition 4.2 *W is a unitary and we have*

$$(\iota \otimes \Delta_A)W = W_{12}W_{13} \text{ and } (\Delta_B \otimes \iota)W = W_{13}W_{23}.$$

This W acts on $A \otimes A$ using the action of B on A in the first leg and left multiplication for the second leg.

Proposition 4.3 *For all $x, x' \in A$ we have*

$$W(x \otimes x') = \Delta(x)(1 \otimes x').$$

We can rewrite the commutation relations between the actions of A and B in terms of the duality W.

Proposition 4.4 *With the same notation as above, we have*

$$W_{23}W_{12} = W_{12}W_{13}W_{23}.$$

Proof. Pair in the first factor with a and in the third factor with b. On the left hand side we get ba. On the right hand side we get

$$\sum_{(a),(b)} a_{(1)} \langle W, a_{(2)} \otimes b_{(1)} \rangle b_{(2)} = \sum_{(a),(b)} a_{(1)} \langle a_{(2)}, b_{(1)} \rangle b_{(2)}$$

□

We see that the Pentagon equation is the same as these commutation rules.

5. An example

Let A be the unital *-algebra generated by a single self-adjoint element h and B the same with a self-adjoint element k. We make this into a dual pair by defining

$$\langle h^n, k^m \rangle = \delta(n, m) n! (it)^n,$$

where δ is the Kronecker delta and t is a given non-zero real number.

It is understood that $h^n = 1$ for $n = 0$ and similarly $k^n = 1$ and $(it)^n = 1$ when $n = 0$.

It is clear that this pairing is non-degenerate.

Proposition 5.1 *The product on A induces a linear map $\Delta : B \to B \otimes B$ satisfying*

$$\langle aa', b \rangle = \langle a \otimes a', \Delta(b) \rangle.$$

The map Δ is given by

$$\Delta(k^n) = \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} (k^j \otimes k^{n-j}).$$

Remark 5.1 Observe that $\Delta(1) = 1 \otimes 1$, $\Delta(k) = k \otimes 1 + 1 \otimes k$ and $\Delta(k^n) = \Delta(k)^n$ for and n. In particular, Δ is a unital *- homomorphism. This situation is very special. It is also coassociative in the sense that $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$.

Consider again this example of the pairing between the two algebras generated by a single self-adjoint element. We have seen that in this case, the coproducts exist with values in the algebraic tensor products. In this case we find (formally)

$$W = \exp \frac{1}{it}(k \otimes h) = \sum_n \frac{1}{n!} \frac{1}{(it)^n} k^n \otimes h^n.$$

In fact, the sum converges in the weak topology on $(A \otimes B)'$.

We see from this formula already an indication how to move to the operator algebraic setting. If h and k are (possibly unbounded) self-adjoint operators, W is a well-defined unitary operator.

Again, it is quite special that W is a unitary. Of course, the pairing is chosen so that this is the case, but it is not obvious that this is possible.

Consider again the example with h and k. We find the following for the action of k on the algebra A of polynomials on h and for the commutation rules.

Proposition 5.2 *We have $k \triangleright (h^n) = (nit)h^{n-1}$. Furthermore $kh = hk + it$.*

Proof.

$$\begin{aligned}\langle h^n, k^{m+1}\rangle &= \delta(n, m+1)n!(it)^n \\ &= (nit)\delta(n-1, m)(it)^{n-1}(n-1)! \\ &= nit\langle h^{n-1}, k^m\rangle.\end{aligned}$$

For the commutation rules we get

$$k \triangleright (hh^n) = (n+1)ith^n = nith^n + ith^n = h(k \triangleright h^n) + ith^n.$$

□

We have considered the two group examples. It is instructive to investigate the pairing further as we did with the example.

We can also consider the Hopf *-algebra A generated by self-adjoint elements a, b with a invertible and $ab = \lambda ba$. Recall that $|\lambda| = 1$. It can be paired with itself:

Proposition 5.3 *Let $z \in \mathbb{C}$ satisfy $z^2 = \lambda$. Then there is a pairing of A with itself given by*

$$\begin{aligned}\langle a, a\rangle &= 1 \qquad & \langle a, b\rangle &= 0 \\ \langle b, a\rangle &= 0 \qquad & \langle b, b\rangle &= i\bar{z}.\end{aligned}$$

The pairing is non-degenerate if and only if λ is not a root of 1.

It is also instructive to complete this example.

6. Further reflections

The usual approach to quantum groups is this:

- Start with a *-algebra A and a coproduct Δ on A.
- Make a set of assumptions and give a name to such pair (A, Δ).
- Construct a dual object.
- And hope it is of the same type.

From what we have seen, it would also be possible to proceed as follows:

- Start with two *-algebras A and B,
- together with a pairing, i.e a non-degenerate bilinear map from $A \times B$ to $\mathbb{C}$.
- How close do you get to a quantum group and its dual?

This is sometimes a relevant (and interesting) approach to study examples.

7. References

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Lecture 2: Multiplier Hopf *-algebras
(6^{th} September 2016)

- Introduction
- Multiplier Hopf algebras
- Existence of the counit and the antipode
- Examples and special cases
- Strong Morita equivalence
- Further reflections and conclusions
- References

8. Introduction (Why multiplier Hopf algebras?)

We have seen that the dual of a finite-dimensional Hopf algebra (A, Δ) is again a Hopf algebra. The coproduct on the dual space is given by the formula $\Delta(f)(a \otimes a') = f(aa')$. We have $\Delta(f) \in A' \otimes A'$ because we can identity $(A \otimes A)'$ with $A' \otimes A'$ as the space is finite-dimensional.

In the infinite-dimensional case, we have a strict inclusion $A' \otimes A' \subsetneq (A \otimes A)'$ and in general, the formula $\Delta(f)(a \otimes a') = f(aa')$ will only define $\Delta(f)$ in $(A \otimes A)'$. We cannot, in general, expect $\Delta(f) \in A' \otimes A'$. This is the reason why we can not make the dual of any Hopf algebra again into a Hopf algebra.

Consider the case of the group algebra $\mathbb{C}G$ of a group G. The dual space of $\mathbb{C}G$ is identified with the space $C(G)$ of all complex functions on G via the pairing $\langle \lambda_p, f \rangle = f(p)$. Recall that we use $p \mapsto \lambda_p$ for the canonical embedding of G in $\mathbb{C}G$.

Because we have $\Delta(\lambda_p) = \lambda_p \otimes \lambda_p$, the coproduct on $C(G)$ would be given by the formula $\Delta(f)(p, q) = f(pq)$. Also here we have a strict embedding $C(G) \otimes C(G) \subsetneq C(G \times G)$ and there is no hope that $\Delta(f) \in C(G \otimes C(G)$ in general.

Denote by $K(G)$ the space of functions with finite support on G. Then we can identity $K(G) \otimes K(G)$ with $K(G \times G)$ and further $C(G \times G)$ with the multiplier algebra $M(K(G \times G))$. So $\Delta(f)$, defined as above, will be an element of the multiplier algebra $M(K(G) \otimes K(G))$.

9. Multiplier Hopf algebras

Let A be an (associative) algebra over $\mathbb{C}$. We do not require the existence of a unit, but we assume that the product is non-degenerate.

Definition 9.1 A multiplier m of A is a pair of linear maps $a \mapsto ma$ and $a \mapsto am$ satisfying $a(mb) = (am)b$ for all $a, b \in A$. We denote the space of multipliers of A by $M(A)$.

We have the following easy, but important result.

Proposition 9.1 *Composition of maps makes $M(A)$ into a unital algebra. It contains A as a dense (essential) two-sided ideal. And it is the largest unital algebra with this property.*

The ideal A is dense because $ma = 0$ for all a implies $m = 0$.

Assume that A and B are two non-degenerate algebras.

Definition 9.2 A homomorphism $\gamma : A \otimes M(B)$ is called non-degenerate if$\gamma(A)B = B$ and $B\gamma(A) = B$.

Proposition 9.2 *If $\gamma : A \otimes M(B)$ is a non-degenerate homomorphism, there is a unique unital extension to a homomorphism (still denoted by γ). It is defined by*

$$\gamma(m)\gamma(a)b = \gamma(ma)b \text{ and } b\gamma(a)\gamma(m) = b\gamma(am)$$

for $a \in A$ and $b \in B$.

There is a slight technical problem. But it is ok if the algebras are idempotent (which is mostly the case).

Definition 9.3 Let A be a non-degenerate algebra. A coproduct on A is a non-degenerate homomorphism $\Delta : A \to M(A \otimes A)$ satisfying coassociativity $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$. If A is a *-algebra, we want Δ to be a *-homomorphism.

We use ι for the identity map and the extensions to $M(A \otimes A)$ of the homomorphisms $\Delta \otimes \iota$ and $\iota \otimes \Delta$, defined first on $A \otimes A$.

Remark 9.1

- Again there is a slight technical problem with this definition. The homomorphisms $\Delta \otimes \iota$ and $\iota \otimes \Delta$ are non-degenerate only if the algebra A is idempotent. There is however a simple 'workaround' if that condition is not fulfilled.
- We need the algebra structure to define a coproduct.

Definition 9.4 Let (A, Δ) be a pair of a non-degenerate algebra with a coproduct. We call it a multiplier Hopf algebra if the maps T_1 and T_2, defined from $A \otimes A$ to $M(A \otimes A)$ by

$$T_1(a \otimes b) = \Delta(a)(1 \otimes b) \text{ and } T_2(c \otimes a) = (c \otimes 1)\Delta(a),$$

are bijective from $A \otimes A$ to $A \otimes A$.

Remark 9.2

- The first requirement is that the maps T_1 and T_2 have range in $A \otimes A$.
- It is an easy consequence of the conditions that the algebra must have local units.
- It also follows automatically that Δ is non-degenerate.

9.1. **Regular multiplier Hopf algebras.**

Definition 9.5 Let (A, Δ) be a multiplier Hopf algebra. If A is a *-algebra and Δ a *-homomorphism, we call(A, Δ) a multiplier Hopf *-algebra.

Definition 9.6 A multiplier Hopf algebra (A, Δ) is called regular if also $(A, \Delta^{\mathrm{cop}})$ satisfies the conditions of a multiplier Hopf algebra.

We use Δ^{cop} for the new coproduct on A obtained by applying the flip map.

Remark 9.3

- Regularity is automatic if A is abelian.
- Regularity is also automatic if A is a multiplier Hopf *-algebra.

9.2. **The motivating example $K(G)$.**

Proposition 9.3 *Let G be a group and $A = K(G)$, the *-algebra of complex functions on G with finite support. We identify $A \otimes A$ with $K(G \times G)$ and $M(A \otimes A)$ with $C(G \times G)$. There is a coproduct on A defined by $\Delta(f)(p, q) = f(pq)$ where $p, q \in G$. The pair (A, Δ) is a multiplier Hopf algebra.*

The proof is easy.

The canonical mapsT_1 and T_2 in this case are given by

$$T_1(g)(p,q) = g(pq,q) \text{ and } T_2(g)(p,q) = g(p,pq)$$

and because G is a group, the maps $(p,q) \mapsto (pq,q)$ and $(p,q) \mapsto (p,pq)$ are bijective.

10. Existence of the Counit and the Antipode

10.1. Existence of the counit.

Proposition 10.1 *Let (A,Δ) be a multiplier Hopf algebra. There exists a unique homomorphism $\varepsilon : A \to \mathbb{C}$ satisfying*

$$(\varepsilon \otimes \iota)(\Delta(a)(1 \otimes b)) = ab$$
$$\textit{and}$$
$$(\iota \otimes \varepsilon)((c \otimes 1)\Delta(a)) = ca$$

for all $a,b,c \in A$.

It is called the counit.

If we have a multiplier Hopf *-algebra, ε is a *-homomorphism.
In the case $K(G)$, we have $\varepsilon(f) = f(e)$ where e is the identity element in G.

10.2. Existence of the antipode.

Proposition 10.2 *Let (A,Δ) be a multiplier Hopf algebra. There exists a unique anti-homomorphism $S : A \to M(A)$ satisfying*

$$m(S \otimes \iota)(\Delta(a)(1 \otimes b)) = \varepsilon(a)b$$
$$m(\iota \otimes S)((c \otimes \iota)\Delta(a)) = \varepsilon(a)c$$

for all $a,b,c \in A$. It flips the coproduct. It is called the antipode.

For a regular multiplier Hopf algebra, the antipode is a bijective map from A to itself. For a multiplier Hopf *-algebra, we have $S(S(a)^*)^* = a$ for all a.
In the case $K(G)$ we have $S(f)(p) = f(p^{-1})$ for all $p \in G$.

Elements of the proof:

Existence of the counit:

- For the counit we must have $(\varepsilon \otimes \iota)T_1(a \otimes b)) = ab$ and therefore $\varepsilon(p)q = mT_1^{-1}(p \otimes q)$.
- This formula will define$\varepsilon(p)$ as a left multiplier. Using T_2 we can show that it is a scalar multiple of 1 and so we can define $\varepsilon : A \to \mathbb{C}$.

Existence of the antipode:

- For the antipode we have $m(S \otimes \iota)T_1(a \otimes b) = \varepsilon(a)b$ and so $S(p)q = (\varepsilon \otimes \iota)T_1^{-1}(p \otimes q)$.
- This defines $S(p)$ as a left multiplier.
- Similarly we define a right multiplier $S'(p)$ by $rS'(p) = (\iota \otimes \varepsilon)T_2^{-1}(r \otimes p)$.
- Then we show that $(rS'(p))q = r(S(p)q)$ by using the fact that the maps $T_2 \otimes \iota$ and $\iota \otimes T_1$ commute.

10.3. Relation with Hopf algebras.

Proposition 10.3 *Any Hopf algebra is a multiplier Hopf algebra. Conversely, if (A,Δ) is a multiplier Hopf algebra, and if A has an identity, then it is a Hopf algebra. If A is finite-dimensional, it will be automatically unital and hence a Hopf algebra.*

We have the following formulas for the inverses of the canonical maps in terms of the antipode.

Proposition 10.4 *If (A,Δ) is a multiplier Hopf algebra with antipode S, then*

$$T_1^{-1}(a \otimes b) = ((\iota \otimes S)\Delta(a))(1 \otimes b)$$
$$T_2^{-1}(c \otimes a) = (c \otimes 1)((S \otimes \iota)\Delta(a))$$

One has to 'cover' these formulas in order to be well-defined.

11. Examples and special cases

11.1. Multiplier Hopf algebras of compact and discrete type.

Definition 11.1 Let (A, Δ) be a multiplier Hopf algebra.

- If A has an identity (i.e. when it is a Hopf algebras) and when it has integrals (see Lecture 3 for this notion), we call it of compact type.
- If there is a non-zero element h in A so that $ah = \varepsilon(a)h$, we call it of discrete type.

An element like h above is called a left cointegral. Similarly, an element k satisfying $ka = \varepsilon(a)k$ for all a is called a right cointegral. Cointegrals can be proven to be unique if they exists.

In the case of a multiplier Hopf *-algebra, with an underlying *-algebra that is an operator algebra, in the first case, we have a compact quantum group and in the second case a discrete quantum group.

11.2. Discrete quantum groups.

Definition 11.2 Let (A, Δ) be a multiplier Hopf *-algebra and assume that$A$ is a direct sum of matrix algebras (with the canonical involution). Then we call it a discrete quantum group.

Proposition 11.1 *Let (A, Δ) be a discrete quantum group. The support of the counit is a component of dimension 1. If h is the identity in this component, it is a self-adjoint projection and $ah = \varepsilon(a)h$. The legs of $\Delta(h)$ are both all of A.*

We will see later (in the next lecture) that the dual of a multiplier Hopf algebra of discrete type is of compact type and vice versa. The same is true for compact and discrete quantum groups. If a multiplier Hopf algebra is both of compact and discrete type, it is finite-dimensional.

11.3. Right module coalgebras. Let (A, Δ) be a multiplier Hopf algebra. Assume that Y is a unital right A-module. The tensor product $Y \otimes Y$ has the two obvious (commuting) right A-module actions.

Definition 11.3 We denote by $Y \overline{\otimes} Y$ the extended module. For elements $z \in Y \overline{\otimes} Y$, the elements

$$z(a \otimes 1) \quad \text{and} \quad z(1 \otimes a)$$

are by definition in $Y \otimes Y$ and

$$(z(a \otimes 1))(1 \otimes b) = (z(1 \otimes b))(a \otimes 1)$$

for all $a, b \in A$.

11.4. Morita A- module coalgebras.

Definition 11.4 A comultiplication Δ on Y is a coassociative linear map from Y to $Y \overline{\otimes} Y$. We call Y a right A-module coalgebra if also $\Delta(ya) = \Delta(y)\Delta(a)$ for all $y \in Y$ and $a \in A$.

Definition 11.5 We call Y a Morita A-module coalgebra if the two canonical maps T and T‘, defined from $Y \otimes A$ to $Y \otimes Y$ by

$$T(y \otimes a) = \Delta(y)(1 \otimes a) \text{ and } T‘(y \otimes a) = \Delta(y)(a \otimes 1),$$

are bijective.

Just as in the case of multiplier Hopf algebras, one can prove the existence of a 'counit' and of an 'antipode':

11.5. **Existence of the counit and the antipode.**

Proposition 11.2 *There exists a linear map $\varepsilon : Y \to \mathbb{C}$ satisfying*

$$(\varepsilon \otimes \iota)(\Delta(y)(1 \otimes a)) = ya \text{ and } (\iota \otimes \varepsilon)(\Delta(y)(a \otimes 1)) = ya$$

for all $y \in Y$ and $a \in A$. Also $\varepsilon(ya) = \varepsilon(y)\varepsilon(a)$.

Proposition 11.3 *For each y, there exists a linear map $S(y) : Y \to A$ satisfying*

$$S(y_{(1)})y_{(2)}a = \varepsilon(y)a \text{ and } y_{(1)}S(y_{(2)})y' = \varepsilon(y)y'.$$

Also $S(ya) = S(a)S(y)$.

The formulas to define the counit ε and the antipode Scome from the inverses of the canonical maps T and T'.

12. Strong Morita equivalence

Define C as the linear span of linear maps from Y to itself of the form $y \mapsto y_1 S(y_2)y$ where $y_1, y_2 \in Y$.

Theorem 12.1 *The space C is a multiplier Hopf algebra and Y is a left Morita C-module coalgebra.*

Definition 12.1 We call a multiplier Hopf algebra (A, Δ) strong Morita equivalent with a multiplier Hopf algebra (C, Δ) if there exists a unital C-A-bimodule Y and a coproduct Δ on Y that makes it into a right Morita A-module and a left Morita C-module coalgebra.

In the finite-dimensional case, we can dualize the concepts and arrive at Galois correspondence. This is still possible in the infinite-dimensional setting when integrals exist.

13. Further reflections and conclusions

We have seen that, from the point of view of duality, the notion of a Hopf algebra is too restrictive. We naturally deal with algebras without identity and a more general notion of a coproduct.

This leads automatically to the theory of multiplier Hopf algebras.

In the next lecture, we will study integrals and we will see that for multiplier Hopf algebras with integrals, there is still the possibility of constructing the dual. The duality extends that of finite-dimensional Hopf algebras to a much bigger class of quantum groups.

Along the same lines, also the concept of Morita equivalence can be dualized in a natural way if integrals exists. This leads to Galois theory for algebraic quantum groups.

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Lecture 3: Algebraic quantum groups and duality
(7^{th} September 2016)

- Introduction
- Integrals on multiplier Hopf algebras
- The dual of an algebraic quantum group
- The duality between discrete and compact quantum groups
- Further reflections and conclusions
- References

15. Introduction

In the first lecture, we mentioned that the dual A' of a finite-dimensional Hopf algebra A is again a Hopf algebra. The result is no longer valid in the infinite-dimensional case because the candidate for the coproduct on A' in general does not map into the tensor product $A' \otimes A'$, but into the bigger space $(A \otimes A)'$.

In the second lecture, we generalized the notion of a coproduct and we introduced multiplier Hopf algebras. Still, in general, we do not expect that for any multiplier Hopf algebra, we will be able to construct the dual.

This will be possible for multiplier Hopf algebras with integrals, the so-called algebraic quantum groups. This is done in this third lecture.

16. Integrals on multiplier Hopf algebras

Let (A, Δ) be a multiplier Hopf algebra.

Definition 16.1 A linear functional φ on A is called *left invariant* if $(\iota\otimes\varphi)((c\otimes 1)\Delta(a)) = \varphi(a)c$ for all $a, c \in A$. Similarly, a linear functional ψ is *right invariant* if $(\psi \otimes \iota)(\Delta(a)(1 \otimes b)) = \psi(a)b$ for all $a, b \in A$. A non-zero left invariant functional is called a *left integral.* A non-zero right invariant functional is called a *right integral.*

In the case of a regular multiplier Hopf algebra, for any $a \in A$ and any linear functional ω we can define the elements

$$(\iota \otimes \omega)\Delta(a) \text{ and } (\omega \otimes \iota)\Delta(a)$$

in $M(A)$. Then $(\iota \otimes \varphi)\Delta(a) = \varphi(a)1$ for a left integral.

16.1. **Integrals on $K(G)$ and $\mathbb{C}G$.** Integrals do not always exist. However, we will see later that they are unique, if they exists. For the two motivating examples associated with a group, they are easy to obtain.

Proposition 16.1 *Let G be a group. The linear map $\varphi : K(G) \to \mathbb{C}$ defined by $\varphi(f) = \sum_p f(p)$ is a left integral. It is also right invariant.*

Indeed, we have

$$((\iota \otimes \varphi)\Delta(f))(p) = \textstyle\sum_q f(pq) = \sum_q f(q).$$

For $\mathbb{C}G$ we define $\varphi(\lambda_e) = 1$ and $\varphi(\lambda_p) = 0$ if $p \neq e$. Here again the left integral is also right invariant.

16.2. **Properties of Integrals.** Assume that (A, Δ) is a regular multiplier Hopf algebra.

Proposition 16.2 *With the above assumption, we have:*

- *If a left integral exists, it is unique up to a scalar. Similarly for a right integral.*
- *If φ is a left integral, then $\varphi \circ S$ is a right integral.*
- *There is an element $\delta \in M(A)$ satisfying $\varphi(S(a)) = \varphi(a\delta)$ for all a.*
- *Integrals are faithful linear functionals.*
- *Integrals admit KMS automorphisms.*
- *There is a scalar τ defined by $\varphi(S^2(a)) = \tau\varphi(a)$*

The various data are related by means of many relations.

Remark 16.1 Before we give elements of the proofs, first some remarks:

- It is clear that we can only have uniqueness up to a scalar. In general, there is no way to choose a particular one.
- Faithfulness for a linear functional ω means that $a \mapsto \omega(\cdot\, a)$ and $a \mapsto \omega(a\, \cdot)$ are injective maps from A to the dual A'.
- If ω is a faithful linear functional on A, we say that it admits a KMS automorphism if there is a linear map $\sigma : A \to A$ so that $\omega(ab) = \omega(b\sigma(a))$ for all a, b.
- A faithful linear functional on a finite-dimensional algebra always admits a KMS automorphism.

  This is not true in the infinite-dimensional case.
- The existence of the scalar τ is a consequence of the uniqueness of left integrals.

Some elements of the proofs

Many of the properties are based on the following result.

Proposition 16.3 *Let φ be a left integral. Let a, b be in A and*

$$c = (\iota \otimes \varphi)((1 \otimes a)\Delta(b)) \text{ and } d = (\iota \otimes \varphi)(\Delta(a)(1 \otimes b)).$$

Then $c = S(d)$. A similar result is valid for a right integral.

Proof. Apply $\iota \otimes \varphi$ to

$$(1 \otimes a)\Delta(b) = \textstyle\sum_{(a)}(S(a_{(1)}) \otimes 1)\Delta(a_{(2)}b)$$

□

Note that in the previous proof, we have used the Sweedler notation.

To prove faithfulness of a left integral φ, assume e.g. that $x \in A$ and that $\varphi(ax) = 0$ for all a. Then

$$(\iota \otimes \varphi)(\Delta(a)(1 \otimes x)) = 0$$

and so also, by the previous result,

$$(\iota \otimes \varphi)((1 \otimes a)\Delta(x)) = 0$$

for all a.

Apply Δ, then with the Sweedler notation, it reads as

$$\textstyle\sum_{(x)} x_{(1)} \otimes x_{(2)}\varphi(ax_{(3)}) = 0$$

for all a.

Replace a by $aS(x_{(2)})$ to obtain

$$(x)x_{(1)}\varphi(aS(x_{(2)})x_{(3)})) = x\varphi(a) = 0,$$

which implies $x = 0$.

Start again with

$$S((\iota \otimes \varphi)(\Delta(a)(1 \otimes b))) = (\iota \otimes \varphi)((1 \otimes a)\Delta(b)).$$

If we apply another left integral φ' and use that $\varphi' \circ S$ is right invariant, we find

$$\varphi'(S(a))\varphi(b) = \varphi(a\delta_b)$$

with $\delta_b = (\varphi' \otimes \iota)\Delta(b)$.

With $\varphi' = \varphi$ and $\varphi(b) = 1$ we find $\delta = \delta_b$.

Also uniqueness of the left integrals will follow from this.

16.3. **Integrals - various relations.** Let (A, Δ) be a regular multiplier Hopf algebra with a left integralφ and a right integral ψ. Denote by σ and σ' the modular automorphisms of φ and ψ respectively. Let δ be the modular element and τ the scaling constant.

Proposition 16.4 *With the same notation as above, we have:*

- $\Delta(\sigma(a)) = (S^2 \otimes \sigma)\Delta(a)$
- $\Delta(\sigma'(a)) = (\sigma' \otimes S^{-2})\Delta(a)$
- $\Delta(S^2(a)) = (\sigma \otimes {\sigma'}^{-1})\Delta(a)$
- $\Delta(\delta) = \delta \otimes \delta$ *and* $S(\delta) = \delta^{-1}$
- $\sigma(\delta) = \tau^{-}\delta$ *and* $\sigma'(\delta) = \tau^{-}\delta$
- $\sigma'(a) = \delta\sigma(a)\delta^{-1}$

We also have $(\varphi \otimes \iota)\Delta(a) = \varphi(a)\delta$ and $(\iota \otimes \psi)\Delta(a) = \psi(a)\delta^{-1}$.

17. The dual

17.1. **The dual of a multiplier Hopf algebra (mha) with integrals.**

Definition 17.1 Let (A, Δ) be a regular multiplier Hopf algebra with a left integral φ. Denote by $\widehat{A}$ the space of linear functionals on A of the form $\varphi(\,\cdot\, a)$ where $a \in A$.

Because of the existence of the modular automorphism and the modular element, elements in $\widehat{A}$ are also of the form $\varphi(a\,\cdot\,)$, $\psi(\,\cdot\, a)$ or $\psi(a\,\cdot\,)$ where ψ is right integral. From the uniqueness of integrals, the set $\widehat{A}$ does not depend on the choice of φ.

Theorem 17.1 *The adjoint of the coproduct on A makes $\widehat{A}$ into a non-degene- rate associative algebra. The adjoint of the product in A provides a coproduct $\widehat{\Delta}$ on $\widehat{A}$. The pair $(\widehat{A}, \widehat{\Delta})$ is again a regular multiplier Hopf algebra with integrals.*

17.2. **The dual - further properties.** Let (A, Δ) be a regular multiplier Hopf algebra with a left integral φ $(\widehat{A}, \widehat{\Delta})$ the dual.

Proposition 17.1 *If $\omega = \varphi(\,\cdot\, a)$, we have $\widehat{\psi}(\omega) = \varepsilon(a)$ for the right integral $\widehat{\psi}$ on the dual.*

With a little manipulation of formulas, we find from this that the dual of $(\widehat{A}, \widehat{\Delta})$ is canonically isomorphic with the original (A, Δ).

Remark 17.1 Regarding the preceding proposition, we have the following:

- We have the modular element, the modular automorphisms and the scaling constant for the dual.
- The dual scaling constant is again τ.
- There are plenty of relations between the objects of (A, Δ) and those of the dual.

17.3. **Radford's formula for S^4.** In order to illustrate the last statement of the previous remark, we mention the following well-known fact:

Proposition 17.2 *We have*
$$S^4(a) = \delta^{-1}(\widehat{\delta} \triangleright a \triangleleft \widehat{\delta}^{-1})\delta$$
for all $a \in A$.

As before, S is the antipode of (A, Δ). There are the modular elements δ and $\widehat{\delta}$ of A and the dual $\widehat{A}$ respectively. Moreover we use the actions of $\widehat{A}$ on A defined by
$$\langle b \triangleright a, b' \rangle = \langle a, b'b \rangle$$
$$\text{and } \langle a \triangleleft b, b' \rangle = \langle a, bb' \rangle,$$
as well as the extensions of these actions to the multiplier algebras (using the same formulas).

We will have an analytical version of this for locally compact quantum groups later!

18. Special cases

18.1. **Duality for discrete and compact type mhas.** Recall that a multiplier Hopf algebra (A, Δ) is called of compact type if A is unital and of discrete type if it has cointegrals.

Proposition 18.1 *If (A, Δ) is of compact type, then $(\widehat{A}, \widehat{\Delta})$ is of discrete type.*

This is obvious. Because $1 \in A$, we have φ and ψ in $\widehat{A}$. And the formula $(\omega \otimes \varphi)\Delta(a) = \omega(1)\varphi(a)$ reads as $bh = \varepsilon(b)h$ for $b = \omega$ and $h = \varphi$ as sitting in $\widehat{A}$.

Similarly $kb = \varepsilon(b)k$ where now $k = \psi$ in $\widehat{A}$.

We also expect the converse. Namely that the dual of a discrete type multiplier Hopf algebra is of compact type (with integrals). However, we first need to show the existence of integrals for a multiplier Hopf algebra of discrete type.

18.2. **Existence of integrals for a discrete type mha.**

Proposition 18.2 *Let (A, Δ) be a multiplier Hopf algebra with a left cointegral h. Then there exists a left integra φ defined by $(\iota \otimes \varphi)\Delta(h) = 1$.*

Proof. i) First we claim that $(1 \otimes a)\Delta(h) = (S(a) \otimes 1)\Delta(h)$.Indeed, the left hand side is
$$\textstyle\sum_{(a)}(S(a_{(1)}) \otimes 1)\Delta(a_{(2)}h) = \sum_{(a)}(\varepsilon(a_{(2)})S(a_{(1)} \otimes 1)\Delta(h).$$

ii) Next we claim that, for any linear functional ω, we have $\omega = 0$ if $(\omega \otimes \iota)\Delta(h) = 0$. Indeed, assume that $(\omega \otimes \iota)\Delta(h) = 0$, then by the formula above, also $(\omega \otimes \iota)((a \otimes 1)\Delta(h)) = 0$ for all a. Apply Δ and S on the second factor and take $aS(h_{(2)})$ for a. □

Now we can define φ by $\varphi(\omega((a \otimes 1)\Delta(h))) = \omega(a)$. It will be well-defined because of ii). An argument as in ii) will yield that all elements in A are of such a form and hence, φ is everywhere defined.

Left invariance of φ will essentially come for free:
$$(\iota \otimes \varphi)\Delta((\omega \otimes \iota)\Delta(h)) = \omega(1)1 = \varphi((\omega \otimes \iota)\Delta(h))$$

Finally, we get the other duality:

Proposition 18.3 *If (A, Δ) is of discrete type, then $(\widehat{A}, \widehat{\Delta})$ is of compact type*

The argument is simple. Because $h \in A$ we have $\varepsilon(a) = \varphi(ah)$ and we see that $\varepsilon \in \widehat{A}$.

19. Conclusions

- We were able to extend the duality of finite-dimensional Hopf algebras (finite quantum groups) to multiplier Hopf algebras with integrals (algebraic quantum groups).
- We find a rich algebraic structure, with many objects that come for free, with a lot of relations among these objects.
- The theory includes compact and discrete quantum groups and more. But not all locally compact quantum groups fit into this framework.
- Nevertheless, all of the algebraic features of locally compact quantum groups, appear already in this setting.

Further steps:

In the next lecture, we will add the involutive structure, work with positive integrals and pass to the Hilbert space level. This is indeed the next intermediate step towards the more general theory of locally compact quantum groups.

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Lecture 4: Towards operator algebraic quantum groups
(9^{th} September 2016)

- Introduction
- The Larson-Sweedler theorem
- Multiplier Hopf algebras with positive integrals
- The associated locally compact quantum group
- Further reflections and conclusions
- References

21. Introduction

In the previous lecture, we studied multiplier Hopf algebras with integrals, the so-called algebraic quantum groups. For these objects, we have a nice duality.

In this lecture, we will focus on algebraic quantum groups and their duality, in the case where the underlying multiplier Hopf algebra is a multiplier Hopf *-algebra, i.e. when A is a *-algebra and Δ a *-homomorphism.

However, we need a 'decent' involutive structure in the sense that the *-algebra is an operator algebra. This means that it can be represented by bounded operators on a Hilbert space and that the adjoint coincides with the involution.

This will be guaranteed if we assume that the integrals are positive.

22. Larson-Sweedler

Recall the following result, known in Hopf algebra theory as the Larson-Sweedler theorem.

Theorem 22.1 *Let (A, Δ) be a pair of a unital algebra with a coproduct that admits a counit. Assume that there is a faithful left integral and a faithful right integral. Then (A, Δ) is a Hopf algebra.*

This is a very important theorem for understanding the operator algebra approach to quantum groups.

We will formulate a stronger result for multiplier Hopf algebras and indicate how it is proven.

We will first consider the general situation and later pass to the involutive case- the main topic for this lecture.

22.1. **Larson-Sweedler for multiplier Hopf algebras.**

Assume that A is a non-degenerate algebra and that Δ is a regular coproduct on A.

Notation 22.1 Recall the notations:

$$T_1(a \otimes b) = \Delta(a)(1 \otimes b) \quad \text{and} \quad T_2(c \otimes a) = (c \otimes 1)\Delta(a)$$
$$T_3(a \otimes b) = (1 \otimes b)\Delta(a) \quad \text{and} \quad T_4(c \otimes a) = \Delta(a)(c \otimes 1).$$

Because we assume Δ regular, these maps all have range in $A \otimes A$. Integrals are defined as before:

Definition 22.1 A *left integral* is a non-zero linear functional φ on A satisfying $(\iota\otimes\varphi)\Delta(a) = \varphi(a)1$ in $M(A)$. A right integral is a non-zero linear functional ψ satisfying $(\psi\otimes\iota)\Delta(a) = \psi(a)1$.

22.2. **Injectivity of the canonical maps.**

Proposition 22.1 *If there is a faithful right integral, the maps T_1 and T_3 are injective. If there is a faithful left integral, the maps T_2 and T_4 are injective.*

Proof. Assume that $\sum_i \Delta(a_i)(1\otimes b_i) = 0$. Multiply with $\Delta(x)$ from the left and apply ψ on the first leg. Then $\sum_i \psi(xa_i)b_i = 0$. This holds for all x and if ψ is faithful, then $\sum_i a_i \otimes b_i = 0$. Hence T_1 is injective.

Similarly for the other 3 cases.

□

22.3. **Surjectivity of the canonical maps.**
The proof of the surjectivity is somewhat more complicated.

Proposition 22.2 *Assume that φ is a left integral. Let $a, b \in A$ and let $p = (\iota\otimes\varphi)(\Delta(a)(1\otimes b))$. Then $p\otimes q$ belongs to the range of T_1 for all q.*

Proof. Take $q \in A$ and define

$$x = (\iota\otimes\iota\otimes\varphi)(\Delta_{13}(a)\Delta_{23}(b)(1\otimes q\otimes 1))$$

(using the leg numbering notation). Then $T_1(x) = p\otimes q$. □

If the left leg of Δ is assumed to be all of A (e.g. when there is a counit) and if φ is faithful, this will imply that T_1 is surjective. The result is still true without the assumption on Δ, but this requires a trick (known as Kustermans trick).

22.4. **The Larson-Sweedler theorem for mhas.**
We have now the following theorem.

Theorem 22.2 *Assume that (A,Δ) is a non-degenerate algebra A with a regular coproduct. If there exists a faithful left integral and a faithful right integral, it is a multiplier Hopf algebra.*

In this case, we can define and characterize the counit $\varepsilon : A \to \mathbb{C}$ by
$$\varepsilon((\iota\otimes\varphi)(\Delta(a)(1\otimes b))) = \varphi(ab)$$
and the antipode as the unique linear map $S : A \to A$ satisfying
$$S((\iota\otimes\varphi)(\Delta(a)(1\otimes b))) = (\iota\otimes\varphi)((1\otimes a)\Delta(b)).$$

23. Positive integrals

23.1. **Multiplier Hopf algebras with positive integrals.** In what follows, we assume that (A,Δ) is a regular multiplier Hopf *-algebra with a positive left integral φ. It can be shown that also the right integral is positive (a non-trivial result).
This is related with the fact that the scaling constant τ is trivial.

Proposition 23.1 *The dual $\widehat{A}$ is again a multiplier Hopf *-algebra with positive integrals.*

The involution in $\widehat{A}$ is defined by $\omega^*(a) = \omega(S(a)^*)^-$ as before.
The right integral on $\widehat{A}$ is defined as $\widehat{\psi}(\omega) = \varepsilon(a)$ if $\omega = \varphi(\,\cdot\, a)$.
We have Plancherel's formula:
If $\omega = \varphi(\,\cdot\, a)$, then $\widehat{\psi}(\omega^*\omega) = \varphi(a^*a)$.

23.2. **Plancherel's formula - Proof.**

Proof. Let $a \in A$ and $\omega = \varphi(\,\cdot\, a)$. Then

$$\begin{aligned}(\omega^*\omega)(x) &= \omega^*(x_{(1)})\omega(x_{(2)}) \\ &= \omega(S(x_{(1)})^*)^-\omega(x_{(2)}) \\ &= \varphi(S(x_{(1)})^*a)^-\varphi(x_{(2)}a) \\ &= \varphi(a^*S(x_{(1)}))\varphi(x_{(2)}a) \\ &= \varphi(a^*a_{(1)})\varphi(xa_{(2)}).\end{aligned}$$

We have used the Sweedler type notation. From the definition of the right integral $\widehat{\psi}$ we find

$$\widehat{\psi}(\omega^*\omega) = \varphi(a^*a_{(1)})\varepsilon(a_{(2)}) = \varphi(a^*a).$$

□

23.3. **The GNS representation for the right integral.**
Let ψ be a positive right integral on a regular multiplier Hopf algebra (A, Δ). Denote the dual by (B, Δ).

Proposition 23.2 *Define $\langle x', x\rangle = \psi(x^*x')$ for $x, x' \in A$. Let $\mathcal{H}$ be the Hilbert space completion and $\Lambda : A \to \mathcal{H}$ the canonical embedding of A in $\mathcal{H}$. There is a non-degenerate *-representation π of A by bounded operators on $\mathcal{H}$ given by $\pi(a)\Lambda(x) = \Lambda(ax)$.*

Proposition 23.3 *There is a non-degenerate *-representation π of B by bounded operators on $\mathcal{H}$ given by (using the brackets for the pairing here).*

$$\pi(b)\Lambda(x) = \textstyle\sum_{(x)} \langle x_{(2)}, b\rangle \Lambda(x_{(1)}).$$

23.4. **Proofs.**
i) Consider the map

$$V : \Lambda(a) \otimes \Lambda(b) \mapsto \textstyle\sum_{(a)} \Lambda(a_{(1)}) \otimes \Lambda(a_{(2)}b)$$

from $\Lambda(A) \otimes \Lambda(A)$ to $\mathcal{H}\overline{\otimes}\mathcal{H}$ (the Hilbert space tensor product). From the right invariance of ψ it follows that this map is isometric:

$$\begin{aligned}\|\textstyle\sum_{(a)} \Lambda(a_{(1)}) \otimes \Lambda(a_{(2)}b)\|^2 &= (\psi \otimes \psi)((1 \otimes b^*)\Delta(a)^*\Delta(a)(1 \otimes b)) \\ &= \psi(a^*a)\psi(b^*b).\end{aligned}$$

ii) The unique continuous extension is a unitary. It is still denoted by V. It is the canonical map T_1 on the Hilbert space level. It is a multiplicative unitary.

iii) Fix a and consider the map $\Lambda(b) \mapsto \sum_{(a)} \Lambda(a_{(1)}) \otimes \Lambda(a_{(2)}b)$. By the previous result, it is bounded. Take the scalar product with $\Lambda(c)$ in the first factor. Then the map $\Lambda(b) \mapsto \Lambda(pb)$ is bounded where now $p = \sum_{(a)} \psi(c^*a_{(1)})a_{(2)}$. Such elements span A. This proves the first proposition.

iv) Now fix b, consider the map $\Lambda(a) \mapsto \sum_{(a)} \Lambda(a_{(1)}) \otimes \Lambda(a_{(2)}b)$. Now take the scalar product with $\Lambda(c)$ in the second factor. Then the map

$$\Lambda(a) \mapsto \textstyle\sum_{(a)} \Lambda(a_{(1)})\psi(c^*a_{(2)}b)$$

is bounded.

As $\psi(c^* \cdot b) = \psi(\cdot b\sigma_\psi(c^*))$ and $A^2 = A$, this proves the second result.

v) The representation of A is a *-representation. This is standard.

vi) Also the representation of B is a *-representation:
Take $a, c \in A$ and $b \in B$. Then

$$\begin{aligned}\langle \pi(b^*)\Lambda(a), \Lambda(c)\rangle &= \psi(c^* a_{(1)})\langle a_{(2)}, b^*\rangle \\ &= \psi(c^* a_{(1)})\langle S(a_{(2)})^*, b\rangle^- \\ &= \psi(c^*_{(1)} a)\langle c_{(2)}, b\rangle^- \\ &= \langle(\Lambda(a), \pi(b)\Lambda(c)\rangle.\end{aligned}$$

Observe that we use the same type of brackets for different things.

23.5. **The associated pair** (M, Δ)**.**

We can now formulate and prove the main result here.

Theorem 23.1 *Let $M = \pi(A)''$. Define $\Delta : M \to M\overline{\otimes}M$ by $\Delta(x) = V(x \otimes 1)V^*$. Then (M, Δ) is a locally compact quantum group.*

In the sense of Kustermans Vaes.

i) Because π is a non-degenerate *-representation of A, we have that $\pi(A)$ is dense in M.
ii) $\Delta(\pi(a)) = (\pi \otimes \pi)(\Delta(a))$ when $a \in A$. Hence Δ is a normal and unital *-homomorphism from M to $M\overline{\otimes}M$. It is still coassociative.
iii) The integrals φ and ψ yield extensions on M to normal semi-finite faithful weights (using left Hilbert algebra theory). They are still invariant (because V is unitary).

23.6. **How to proceed?**

We can now proceed with constructing the objects, like the modular automorphisms, the antipode, ... on the Hilbert space level. The modular automorphisms come for free from the theory of weights on von Neumann algebras. The antipode and its polar decomposition follow from the following result.

Proposition 23.4 *The map $\Lambda(a) \mapsto \Lambda(S(a^*))$ is preclosed. Its closure K is a conjugate linear involutive operator satisfying $K\pi(a)K = \pi(S(a^*))$ for all $a \in A$.*

Proof. Take $a, c \in A$. Then

$$\begin{aligned}\langle \Lambda(S(a^*)), \Lambda(c)\rangle &= \psi(c^* S(a^*)) = \varphi(a^* S(c)^*) \\ &= \psi(a^* S(c)^* \delta^{-1}) = \langle \Lambda(S(c)^*\delta^{-1}), \Lambda(a)\rangle\end{aligned}$$

□

We also have the closure of the map $T : \Lambda(a) \mapsto \Lambda(a^*)$. We have the polar decompositions of these two closed maps

$$T = J\nabla^{\frac{1}{2}} \quad \text{and} \quad K = IN^{\frac{1}{2}}.$$

The algebraic relations we had all have their counter part with operators on the Hilbert space.

An important relation is

$$(T \otimes K)V = V^*(T \otimes K),$$

combined with the uniqueness of polar decompositions:

$$(\nabla^{it} \otimes N^{it})V(\nabla^{-it} \otimes N^{-it}) = V \text{ and } (J \otimes I)V(J \otimes I) = V^*.$$

The dual von Neumann algebra $\widehat{M}$ is obtained as $\pi(B)''$ and the dual coproduct $\widehat{\Delta}$ on $\widehat{M}$ is given by $y \mapsto V^*(1 \otimes y)V$, (combined with the flip map). It extends the coproduct on B.

We have the analytical versions of the dual objects. Also the various algebraic relations have a Hilbert space level form.

In fact, the analytical structure is also realized on the purely algebraic level (which is a bit remarkable). This was first proven by J. Kustermans.

The only drawback is that the scaling constant τ is trivial for these cases. This is not so for general locally compact quantum groups.

23.7. **Radford's formula again.**

Recall the formula

$$S^4(a) = \delta^{-1}(\widehat{\delta} \triangleright a \triangleleft \widehat{\delta}^{-1})\delta$$

from the previous lecture. If we translate this to the operator algebraic setting, we find

$$P^{-2it} = \delta^{it}\,(J\delta^{it}J)\,\widehat{\delta}^{it}\,(\widehat{J}\widehat{\delta}^{it}\widehat{J}).$$

In this formula, P is an operator that implements the square of the antipode. And J and $\widehat{J}$ are the modular conjugations.

24. Conclusions

In the previous lecture, we studied multiplier Hopf algebras with integrals. And we obtained a dual of the same type.

In this lecture, we considered multiplier Hopf *-algebras with positive integrals. Using the GNS representation associated with the right integral, we could lift this algebraic quantum group on a Hilbert space level to a locally compact quantum group.

All the objects give rise to Hilbert space representations and the relations among these objects are reflected by similar relations of these Hilbert space operators (cf. e.g. Radford's formula.

There are different approaches.

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Lecture 5: Algebraic quantum groupoids
(13^{th} September 2016)

- Introduction
- Weak multiplier Hopf algebras
- Separability idempotents
- Algebraic quantum groupoids
- An example and its dual
- Further reflections and conclusions
- References

26. Introduction

In the previous 4 lectures, we developed the theory of algebraic quantum groups (multiplier Hopf algebras with integrals).

In the 4th lecture, we discussed the case where the underlying algebra is a *-algebra, and the integrals positive linear functionals. In that context, we were able to lift the structure to a Hilbert space level and by doing so, associate a locally compact quantum group.

In this last lecture of my series, I want to present some aspects of the theory of quantum groupoids.

The process starting with finite quantum groupoids (finite-dimensional weak Hopf algebras) via weak multiplier Hopf algebras, weak multiplier Hopf algebras with integrals (algebraic quantum groupoids) over the involutive case with positive integrals, all the way up to locally compact quantum groupoids, is very similar as for quantum groups.

A groupoid is a set G with a distinguished subset of $G\times G$. It is by definition the set of pairs (p,q) for which the product pq is defined. Further, the product is associative in an obvious sense. There is now a set of units G_0, sitting in G and a source and a target map $s,t : G\to G_0$. The product pq is defined only if the source $s(p)$ is the same as the target $t(q)$. Finally each element p as a unique inverse p^{-1} with the property that

$$p^{-1}p = s(p) \text{ and } pp^{-1} = t(p).$$

Example 26.1 Take a set X and let $G = X\times X$. Define the product of elements $p=(x,y)$ and $q=(x',y')$ only if $y=x'$ and then $pq=(x,y')$. Units are elements (x,x), and the inverse of (x,y) is (y,x). Further we have

$$s((x,y)) = (y,y) \text{ and } t((x,y)) = (x,x).$$

27. Weak multiplier Hopf algebras

27.1. **Weak multiplier Hopf algebras associated with G.** Let G be a groupoid. The algebra A of complex functions with finite support in G is a weak multiplier Hopf algebra if we define the coproduct as

$$\Delta(f)(p,q) = \begin{cases} f(pq) & \text{if } pq \text{ is defined,} \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, we can also consider the groupoid algebra $\mathbb{C}G$ of G. Now the coproduct is defined by $\Delta(\lambda_p) = \lambda_p\otimes\lambda_p$ where $p\mapsto\lambda_p$denotes the embedding of G in the groupoid algebra $\mathbb{C}G$. This is again a weak multiplier Hopf algebra.

The first algebra has no identity if G is infinite, while the second one has no identity as soon as the set of units is infinite.

The second weak multiplier Hopf algebra is dual to the first one.

Example 27.1 Let X be any set. Consider the example with the Cartesian product $G := X \times X$. Recall that the product of two elements $p = (x, y)$ and $q = (x', y')$ is defined if $y = x'$ and then $pq = (x, y')$.

The algebra A is $K(X \times X)$ and the coproduct is

$$\Delta(f)(x, y; x', y') = \begin{cases} f(x, y') & \text{if y=x'}, \\ 0 & \text{otherwise.} \end{cases}$$

For this groupoid, the groupoid algebra $\mathbb{C}G$ is again the space $K(X \times X)$ but now the product is

$$(fg)(x, y) = \sum_u f(x, u)g(u, y).$$

Here, the coproduct is given by the formula

$$\Delta(f)(x, y; x', y') = \begin{cases} f(x, y) & \text{if } x = x' \text{ and } y = y', \\ 0 & \text{otherwise.} \end{cases}$$

For the first algebra, the identity (in the multiplier algebra) is the constant function with value 1 on $X \times X$. The canonical idempotent E (which is eventually $\Delta(1)$) is the function taking the value 1 in points $(x, u; u, y)$ and 0 in other points.

For the second algebra, the identity is the function with value 1 on elements (x, x) and 0 in other points. The canonical idempotent now is the function that has the value 1 only in points of the form $(u, u; u, u)$.

We will give a full characterization of the concept and its duality. We illustrate the theory with a non-commutative version of this example.

28. Separability idempotents

Assume that B and C are non-degenerate idempotent algebras.

Definition 28.1 Let E be an idempotent in $M(B \otimes C)$. Assume that

- $(b \otimes 1)E$ and $E(1 \otimes c)$ belong to $B \otimes C$,
- the left leg of E is all of B and the right leg of E is all of C,
- there exist non-degenerate anti-homomorphisms $S_B : B \to M(C)$ and $S_C : C \to M(B)$ satisfying

$$E(b \otimes 1) = E(1 \otimes S_B(b)) \text{ and } (1 \otimes c)E = (S_C(c) \otimes 1)E.$$

Then E is called a separability idempotent.

If the anti-homomorphisms map into C and B respectively, then E is called semi-regular. If moreover they are anti-isomorphisms, E is called regular.

Example 28.1 *(The standard abelian example)*
For the case where the underlying algebras B and C are abelian, the standard example is the following.

Let X be a set. Let B and C be the algebra $K(X)$. So $M(B \otimes C)$ is the algebra $C(X \times X)$ of all complex functions on the Cartesian product $X \times X$. Define E as the function taking the value 1 in points (x, x) and 0 in all other points. This is a regular separability idempotent.

The antipodal maps S_B and S_C are just the identity maps:

$$(E(1 \otimes f))(x, y) = E(x, y)f(y) = E(x, y)f(x) = (E(f \otimes 1))(x, y)$$

for all x, y. We use that $E(x, y)$ is only non-zero if $x = y$.

Example 28.2 *(The standard non-abelian example)*
In the case where the algebras B and C are non-abelian, the following example is a basic building block.

For B and C we take the algebra $M_n(\mathbb{C})$ of $n \times n$ matrices over the complex numbers. We denote a set of matrix elements by (e_{ij}) where t $i, j = 1, 2, \ldots, n$. Then we have the idempotent

$$E_0 = \frac{1}{n} \sum_{i,j=1}^{n} e_{ij} \otimes e_{ij}$$

in $B \otimes C$. The anti-isomorphisms S_B and S_C are the transposition of matrices, given by $S_0 : e_{ij} \mapsto e_{ji}$ for all i, j. In this case S_B and S_C are each others inverses. This is also a regular separability idempotent.

We can modify the previous example.

Proposition 28.1 *Let $n \in \mathbb{N}$ and take again for B and C the algebra $M_n(\mathbb{C})$ as before. Let r and s be any two invertible matrices in $M_n(\mathbb{C})$ and assume that $Tr(sr) = n$ where Tr is the trace on $M_n(\mathbb{C})$, normalized so that $Tr(1) = n$. Put*

$$E = (r \otimes 1)E_0(s \otimes 1)$$

where E_0 is as before. Then E is a regular separability idempotent. The antipodal maps are given by

$$S_B(b) = S_0(sbs^{-1}) \quad \textit{and} \quad S_C(c) = rS_0(c)r^{-1}$$

for all b in B and c in C where S_0 is transposition of matrices.

We have the following main properties:

Proposition 28.2 *There exist unique linear functionals φ_B on B and φ_C on C so that*

$$(\varphi_B \otimes \iota)E = 1 \textit{ and } (\iota \otimes \varphi_C)E = 1.$$

28.1. **The distinguished linear functionals.** In the regular case, these functionals have nicer properties:

- They are faithful, i.e. the maps $b \mapsto \varphi_B(\,\cdot\, b)$ and $b \mapsto \varphi_B(b\,\cdot\,)$ are injective and similarly for φ_C.
- They have KMS-automorphisms σ_B of B and σ_C of C satisfying

$$\varphi_B(bb') = \varphi_B(b'\sigma_B(b)) \textit{ and } \varphi_C(cc') = \varphi_C(c'\sigma_C(c)).$$

It is not hard to find the formulas in the case of our examples.

29. Algebraic quantum groupoids

Let A be a non-degenerate algebra and assume that $\Delta : A \to M(A \otimes A)$ is a coproduct on A. We assume that

$$\Delta(a)(1 \otimes b) \quad \textit{and} \quad (c \otimes 1)\Delta(a)$$

are in $A \otimes A$. We assume that Δ is weakly non-degenerate, i.e. that there is an idempotent $E \in M(A \otimes A)$ satisfying

$$\Delta(A)(A \otimes A) = E(A \otimes A) \quad \textit{and} \quad (A \otimes A)\Delta(A) = (A \otimes A)E.$$

Then Δ has a unique extension to a homomorphism from $M(A)$ to $M(A \otimes A)$ satisfying $\Delta(1) = E$. Similarly we can extend $\Delta \otimes \iota$ and $\iota \otimes \Delta$ and coassociativity is now written as

$$(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta.$$

We assume that

$$(\Delta \otimes \iota)E = (1 \otimes E)(E \otimes 1) = (E \otimes 1)(1 \otimes E).$$

The next assumptions are as follows.

Assumption. We assume that there exist subalgebras B and C of $M(A)$, such that $BA = AB = A$ and $CA = AC = A$.

Then these subalgebras are non-degenerate and their multiplier algebras embed in $M(A)$. The same is true for the multiplier algebra of $B \otimes C$.

Assumption. Now we require moreover that E is in $M(B \otimes C)$ and that it is a separability idempotent.

It is not hard to show that the algebras B and C are completely determined as the left and the right leg of E as sitting in $M(A \otimes A)$. We just say that E is a separability idempotent.

Example 29.1 Start with a separability idempotent $E \in M(B \otimes C)$. Define $A = C \otimes B$ and $\Delta : A \to M(A \otimes A)$ by

$$\Delta(c \otimes b) = c \otimes E \otimes b$$

for $b \in B$ and $c \in C$.

We will use this example throughout the rest of this lecture.

Consider the special case with $B = C = K(X)$ so that $A = K(X \times X)$. Recall that $E(x, y) = 1$ if $x = y$ and 0 otherwise. Then Δ is

$$\Delta(f)(x, y; x', y') = \begin{cases} f(x, y') & \text{if } y = x', \\ 0 & \text{otherwise.} \end{cases}$$

29.1. **Integral.** Let (A, Δ) be a non-degenerate algebra with a weakly non-degenerate regular and full coproduct so that the canonical idempotent E is a regular separability idempotent with legs B and C in $M(A)$.

Definition 29.1 A left integral is a linear functional φ on A such that $(\iota \otimes \varphi)\Delta(a) \in M(C)$ for all $a \in A$. A right integral is a linear functional ψ on A such that $(\psi \otimes \iota)\Delta(a) \in M(B)$ for all $a \in A$.

Definition 29.2 We call (A, Δ) an algebraic quantum groupoid if there exists a faithful set of left integrals and a faithful set of right integrals.

Recall the Larson-Sweedler theorem from the previous lecture:

Theorem 29.1 *Let A be a unital algebra and Δ a coproduct on A with a counit. If A has a faithful left integral and a faithful right integral, then it is a Hopf algebra.*

We generalized this result to the case of multiplier Hopf algebras (with Shuanhong Wang) and further to weak multiplier Hopf algebras (with Byung-Jay Kahng):

Theorem 29.2 *If (A, Δ) is an algebraic quantum groupoid, then it is a regular weak multiplier Hopf algebra. In particular, there exists a counit and an antipode.*

30. Example

30.1. **An example of an algebraic quantum groupoid.** Consider a regular separability idempotent $E \in M(B \otimes C)$. Let $A = C \otimes B$ and $\Delta(c \otimes b) = c \otimes E \otimes b$ as before.

Theorem 30.1 *The linear functional $\varphi : c \otimes b \mapsto \varphi_C(c)\varphi_B(b)$ is a left and a right integral. It is faithful.*

Proof. We have (using a Sweedler type notation $E = E_{(1)} \otimes E_{(2)}$)

$$\begin{aligned}(\iota \otimes \varphi)(\Delta(c \otimes b)) &= \varphi_C(E_{(2)})\varphi_B(b)\, c \otimes E_{(1)} \\ &= \varphi_B(b)\, c \otimes 1.\end{aligned}$$

This belongs to C, viewed as sitting in $M(A)$. Hence it is a left integral. A similar argument gives that it is also a right integral. Faithfulness follows from the faithfulness of φ_C and φ_B. $\square$

For the counit we get the following result.

Proposition 30.1 *The counit is given by*

Proof.

$$\varepsilon(c \otimes b) = \varphi_B(S_C(c)b) = \varphi_C(cS_B(b)).$$

$$\begin{aligned}(\iota \otimes \varepsilon)\Delta(c \otimes b) &= \varphi_C(E_{(2)}S_B(b))c \otimes E_{(1)} \\ &= \varphi_C(E_{(2)})c \otimes E_{(1)}b \\ &= c \otimes b.\end{aligned}$$

Similarly $(\varepsilon \otimes \iota)\Delta(c \otimes b) = c \otimes b$ for all b and c. □

For the antipode we have $S(c \otimes b) = S_B(b) \otimes S_C(c)$. This is expected, but it is harder to show.

30.2. **The dual for this example.** We will not give a general treatment of the dual. We will use the example.

Proposition 30.2 *There is a non-degenerate pairing of A with $B \otimes C$ given by*

$$\langle c \otimes b, u \otimes v\rangle = \varphi_B(bS_C(v))\, \varphi_C(S_B(u)c)).$$

This defines a bijective map $u \otimes v \mapsto \langle\,\cdot\,, u \otimes v\rangle$ from $B \otimes C$ to the dual $\widehat{A}$ of A.

We will use $u \diamond v$ when we consider $u \otimes v$ as sitting in the dual space $\widehat{A}$. The coproduct on A yields a product on $B \otimes C$ by the formula

$$\langle c \otimes b, (u \diamond v)(u' \diamond v')\rangle = \langle \Delta(c \otimes b), (u \diamond v) \otimes (u' \diamond v')\rangle.$$

30.3. **The algebra $B \diamond C$.**

Proposition 30.3 *The product is given by $(u \diamond v)(u' \diamond v') = \varepsilon(v \otimes u')\, u \diamond v'$*

Proof.

$$\begin{aligned}\langle c{\otimes}b, (u \diamond v)(u' \diamond v')\rangle &= \langle c \otimes E \otimes b, (u \diamond v) \otimes (u' \diamond v')\rangle \\ &= \langle c \otimes E_{(1)}, u \diamond v\rangle\langle E_{(2)} \otimes b, u' \diamond v'\rangle \\ &= \varphi_B(E_{(1)}S_C(v))\varphi_C(S_B(u)c)\varphi_B(bS_C(v'))\varphi_C(S_B(u')E_{(2)}) \\ &= \varphi_B(E_{(1)})\varphi_C(S_B(u)c)\varphi_B(bS_C(v'))\varphi_C(S_B(u')E_{(2)}S_BS_C(v)) \\ &= \varphi_C(S_B(u)c)\varphi_B(bS_C(v'))\varphi_C(S_B(u')S_BS_C(v)) \\ &= \varphi_C(S_B(u')S_BS_C(v))\, \langle c \otimes b, u \diamond v'\rangle \\ &= \varphi_B(S_C(v)u')\langle c \otimes b, u \diamond v'\rangle \\ &= \varepsilon(v \otimes u')\langle c \otimes b, u \diamond v'\rangle.\end{aligned}$$

□

30.4. **The coproduct on $B \diamond C$.** The coproduct on $B \diamond C$ is 'defined' by the formula

$$\langle (c \otimes b) \otimes (c' \otimes b'), \Delta(u \diamond v)\rangle = \langle cc' \otimes bb', u \diamond v\rangle.$$

Proposition 30.4 *The coproduct on $B \diamond C$ is*

$$\Delta(u \diamond v) = \sum_{(u),(v)} (u_{(1)} \diamond v_{(1)}) \otimes (u_{(2)} \diamond v_{(2)})$$

where we use the Sweedler notations

$$\begin{aligned}\textstyle\sum_{(u)} u_{(1)} \otimes u_{(2)} &= \Delta_B(u) = ((\iota \otimes S_C)E)(1 \otimes u) \\ \textstyle\sum_{(v)} v_{(1)} \otimes v_{(2)} &= \Delta_C(v) = (v \otimes 1)((S_B \otimes \iota)E).\end{aligned}$$

The coproducts are known by the algebraists.

30.5. **Special case.** If we again consider the special case with $B = C = K(X)$. The dual algebra is $K(X \times X)$ with the product

$$(fg)(x, y) = \sum_u f(x, u)g(u, y)$$

and the coproduct is

$$\Delta(f)(x, y; x', y') = \begin{cases} f(x, y) & \text{if } x = x' \text{ and } y = y', \\ 0 & \text{otherwise.} \end{cases}$$

This is as in the introduction.

It is useful to consider the case where E is build with two invertible matrices and see what we get then.

Conclusions: We have used an example to illustrate the construction of the dual for an algebraic quantum groupoid.

- The starting point is a separability idempotent E in $M(B \otimes C)$.
- The algebra A is $C \otimes B$.
- The coproduct on A is $\Delta(c \otimes b) = c \otimes E \otimes b$.
- The dual algebra $\widehat{A}$ is like a matrix algebra.
- A special case gives the dual pair of weak multiplier Hopf algebras associated to the trivial groupoid $X \times X$.

These examples help to understand the duality of regular weak multiplier Hopf algebras with integrals (algebraic quantum groupoids). The general theory however is more complex.

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Multiplicative unitaries and locally compact quantum groups

Lectures by S. L. Woronowicz

Lecture 1: Multiplicative unitaries and locally compact quantum groups
(8^{th} September 2016)

32. Introduction

In the last 25 years, multiplicative unitary operators proved to be one of the main tool in the theory of locally compact quantum groups. In this series of five lectures, we will formulate the theory of manageable multiplicative unitaries and show how they produce quantum groups. We are going to present complete (and in most cases new) proofs. In the original version of this theory an important role was played by Hilbert Schmidt operators. Now we shall use derived pentagon equations that make the proofs much simpler. Next we plan to discuss more recent developments. These are homomorphisms of quantum groups, actions of quantum groups on C^* algebras, crossed products and if time permits Landstad-Vaes theory.

33. Multiplicative unitaries (Baaj-Skandalis)

Let W be a bounded operator acting on $H \otimes H$: $W \in \mathrm{B}(H \otimes H)$. We shall use the leg numbering notation: $W_{12} = W \otimes I$, $W_{13} = (I \otimes \Sigma)(W \otimes I)(I \otimes \Sigma)$ and $W_{23} = I \otimes W$. In these formulae I is the unit operator acting on H and $\Sigma \in \mathrm{B}(H \otimes H)$ is the flip: $\Sigma(x \otimes y) = y \otimes x$ for any $x, y \in H$. Clearly $W_{12}, W_{13}, W_{23} \in \mathrm{B}(H \otimes H \otimes H)$.

Definition 33.1 Let H be a Hilbert space and W be a unitary operator acting on $H \otimes H$. We say that W is a multiplicative unitary if it satisfies the pentagonal equation

$$W_{23}W_{12} = W_{12}W_{13}W_{23}.$$

We shall consider unitary operators $X, Y, Z, \dots$ acting on tensor product of Hilbert spaces.

Definition 33.2 Let H_1, H_2, H_3 be Hilbert spaces and $X \in \mathrm{B}(H_1 \otimes H_2)$ and $Y \in \mathrm{B}(H_2 \otimes H_3)$ be unitary operators. We say that pair (X, Y) is $\#$-composable if operator $X_{12}^* Y_{23} X_{12} Y_{23}^* \in \mathrm{B}(H_1 \otimes H_2 \otimes H_3)$ has trivial second leg. Then $X_{12}^* Y_{23} X_{12} Y_{23}^* = Z_{13}$, where Z is a unitary operator acting on $H_1 \otimes H_3$. In what follows, we shall write

$$Z = X \# Y.$$

Remark 33.1 The $\#$-composition is present (implicitly) in many formulae of the theory of locally compact quantum groups. The reader should notice that a unitary W is multiplicative unitary if and only if (W, W) is $\#$-composable and $W = W \# W$. Similarly V is adapted to W if and only if (V, W) is $\#$-composable and $V = V \# W$. In general pentagon equations considered in the theory are of the form $Z = X \# Y$.

In certain cases the $\#$-composition is associative.

Definition 33.3 Let H_2, H_3 be Hilbert spaces and $Y \in \mathrm{B}(H_2 \otimes H_3)$ be a unitary operator.We say that Y is flipfree if for any $n \in \mathrm{B}(H_2)$ and $m \in \mathrm{B}(H_3)$, relation $(I \otimes m)Y = Y(n \otimes I)$ implies that n and m are multiples of I.

The flip operator $\Sigma \in \mathrm{B}(H \otimes H)$ is not flipfree. Indeed for any $n \in \mathrm{B}(H)$ we have $(I \otimes n)\Sigma = \Sigma(n \otimes I)$. It turns out that manageable operators are flipfree.

Proposition 33.4 Let H_1, H_2, H_3, H_4 be Hilbert spaces and $X \in \mathrm{B}(H_1 \otimes H_2)$, $Y \in \mathrm{B}(H_2 \otimes H_3)$ and $Z \in \mathrm{B}(H_3 \otimes H_4)$ be unitary operators. Assume that (X, Y) and (Y, Z) are #-composable and that Y is flipfree. Then $(X\#Y, Z)$ and $(X, Y\#Z)$ are #-composable and

$$(X\#Y)\#Z = X\#(Y\#Z).$$

Before the proof of the above proposition, we make the following remark:

Remark 33.2 Instead of the flipfreeness of Y one may assume that one of the pairs $(X\#Y, Z)$, $(X, Y\#Z)$ is #-composable. Then the other pair is #-composable and associativity holds.

$$(X\#Y)\#Z = X\#(Y\#Z).$$

Proof. We shall consider two unitaries:

$$R = (X\#Y)^*_{13} Z_{34} (X\#Y)_{13} Z^*_{34},$$
$$S = X^*_{12} (Y\#Z)_{24} X_{12} (Y\#Z)^*_{24}.$$

They act on $H_1 \otimes H_2 \otimes H_3 \otimes H_4$. The reader should notice that R has trivial second leg and that S has trivial third leg. It is sufficient to show that $R = S$. Expanding #, one can easily verify that

$$RY_{23} = Y_{23}S.$$

Remembering that Y is flipfree we conclude that the third leg of R and the second leg of S are also trivial. Therefore Y_{23} commutes with R (and S) and R=S. The statement follows. $\square$

34. Category of Locally Compact Quantum Groups

We introduce the category of locally compact quantum groups (lcqg) as follows:

Morphisms: Unitaries acting on tensor product of two Hilbert spaces. And composition of morphisms is given by #-composition.

Objects: Flipfree multiplicative unitaries.

Let $W \in \mathrm{B}(H \otimes H)$ and $V \in \mathrm{B}(K \otimes K)$ be flipfree multiplicative unitaries and $X \in \mathrm{B}(K \otimes H)$. We say that X is a morphism from W to V if

$$X\#W = V\#X = X.$$

If X is a morphism from W $(X\#W = X)$ and Y is a morphism to W $(W\#Y = Y)$, then $(X\#W)\#Y = X\#(W\#Y)$ and (X, Y) is #-composable.

34.1. **Transposition map.**

Let H be a separable Hilbert space and $\overline{H}$ be the complex conjugate of H. For any $x \in H$, the corresponding element of $\overline{H}$ will be denoted by $\overline{x}$. Then $H \ni x \to \overline{x} \in \overline{H}$ is an antiunitary map. In particular $(\overline{x}|\overline{y}) = (y|x)$ for any $x, y \in H$. For any closed operator m acting on H, $m^\top$ will denote the transpose of m. By definition $D(m^\top) = \overline{D(m^*)}$ and

$$m^\top \overline{x} = \overline{m^* x}$$

for any $x \in D(m^*)$. For any $m \in \mathrm{B}(H)$, $m^\top$ is a bounded operator acting on $\overline{H}$ such that $\left(\overline{x}|m^\top|\overline{y}\right) = (y|mx)$ for all $x, y \in H$. Clearly $\mathrm{B}(H) \ni m \to m^\top \in \mathrm{B}(\overline{H})$ is an antiisomorphism of C^*-algebras. Setting $\overline{\overline{x}} = x$ we identify $\overline{\overline{H}}$ with H. With this identification $m^{\top\top} = m$ for any $m \in \mathrm{B}(H)$.

34.2. **Manageability.**

Definition 34.1 Let H be a Hilbert space and $W \in \mathrm{B}(H \otimes H)$ be a unitary operator. We say that W is manageable if there exist a positive selfadjoint operator Q acting on H and a unitary operator $\widetilde{W}$ acting on $\overline{H} \otimes H$ such that $\ker(Q) = \{0\}$,

$$W^*(Q \otimes Q)W = Q \otimes Q$$

and

$$(x \otimes u|W|z \otimes y) = \left(\overline{z} \otimes Qu\middle|\widetilde{W}\middle|\overline{x} \otimes Q^{-1}y\right)$$

for any $x, z \in H$, $y \in D(Q^{-1})$ and $u \in D(Q)$.

34.3. **Duality.**

Let

$$\widehat{W} = \Sigma W^* \Sigma.$$

$$\left\{ \begin{array}{c} W \text{ is a manageable} \\ \text{multiplicative unitary} \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{c} \widehat{W} \text{ is a manageable} \\ \text{multiplicative unitary} \end{array} \right\}$$

Operators Q are the same for W and $\widehat{W}$.

Manageability $\Longrightarrow$ flipfreeness.

Proposition 34.2 Let $W \in \mathrm{B}(H \otimes H)$ be a manageable multiplicative unitary. Then W is flipfree.

Proof. Let $m, n \in \mathrm{B}(H)$. Assume that $(I \otimes m)W = W(n \otimes I)$. Then for any for any $x, y \in D(Q^{-1})$ and $u, z \in D(Q)$ we have

$$(x \otimes m^*u|W|z \otimes y) = (x \otimes u|W|nz \otimes y)\,,$$

$$\left(\overline{z} \otimes Qm^*u\middle|\widetilde{W}\middle|\overline{x} \otimes Q^{-1}y\right) = \left(\overline{nz} \otimes Qu\middle|\widetilde{W}\middle|\overline{x} \otimes Q^{-1}y\right),$$

$$\overline{z} \otimes Qm^*u = \overline{nz} \otimes Qu.$$

□

34.4. **The main theorem.**

Let H be a separable Hilbert space, $W \in \mathrm{B}(H \otimes H)$ be a manageable multiplicative unitary and

$$A = \left\{(\omega \otimes \mathrm{id})W : \omega \in \mathrm{B}(H)_*\right\}^{\text{norm closure}},$$

$$\widehat{A} = \left\{(\mathrm{id} \otimes \omega)(W^*) : \omega \in \mathrm{B}(H)_*\right\}^{\text{norm closure}}.$$

Then

1. A and $\widehat{A}$ are separable C^*-algebras acting on H in a non-degenerate way.

2. $W \in \mathrm{M}(\widehat{A} \otimes A)$.

3. There exists unique $\Delta \in \mathrm{Mor}(A, A \otimes A)$ such that

$$(\mathrm{id} \otimes \Delta)W = W_{12}W_{13}.$$

Δ is coassociative: $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$ and $\{\Delta(a)(I \otimes b) : a, b \in A\}$ and $\{(a \otimes I)\Delta(b) : a, b \in A\}$ are linearly dense subsets of $A \otimes A$.

4. There exists a unique closed linear operator κ acting on A such that $\{(\omega\otimes\mathrm{id})W : \omega\in \mathrm{B}(H)_*\}$ is a core (essential domain) of κ and

$$\kappa\Big((\omega\otimes\mathrm{id})W\Big)=(\omega\otimes\mathrm{id})(W^*)$$

for any $\omega\in\mathrm{B}(H)_*$. The domain $D(\kappa)$ is a subalgebra of A and κ is antimultiplicative: for any $a,b\in D(\kappa)$, $\kappa(ab)=\kappa(b)\kappa(a)$. The image $\kappa(D(\kappa))$ coincides with $D(\kappa)^*$ and $\kappa(\kappa(a)^*)^*=a$ for any $a\in D(\kappa)$.

The operator κ admits the following polar decomposition:

$$\kappa = R\circ\tau_{i/2}$$

where $\tau_{i/2}$ is the analytic generator of a one parameter group $(\tau_t)_{t\in\mathbb{R}}$ of *-automorphisms of the C^*-algebra A and R is an involutive normal antiautomorphism of A commuting with automorphisms τ_t for all $t\in\mathbb{R}$. In particular $D(\kappa)=D(\tau_{i/2})$. R and $(\tau_t)_{t\in\mathbb{R}}$ are uniquely determined. τ is called scaling group and R is called unitary antipode. We shall write a^R instead of $R(a)$. This is the same notation as the one used for transposition: traditionally we write $a^\top$ instead of $\top(a)$.

For any $a\in A$ and $t\in\mathbb{R}$ we have

$$\Delta(\tau_t(a))=(\tau_t\otimes\tau_t)\Delta(a),$$
$$\Delta(a^R)=\mathrm{flip}\left(\Delta(a)^{R\otimes R}\right).$$

5. Let $\widetilde{W}$ and Q be the operators given in definition 2.1. Then for any $t\in\mathbb{R}$ and $a\in A$ we have:

$$\tau_t(a)=Q^{2it}aQ^{-2it}.$$

Moreover we have:

$$W^{\top\otimes R}=\widetilde{W}^*.$$

6. Denoting by $\widehat{\tau}$ and $\widehat{R}$ the scaling group and the unitary antipode related to $\widehat{W}$ we have

$$(\widehat{\tau}_t\otimes\tau_t)W=W,$$
$$W^{\widehat{R}\otimes R}=W.$$

34.5. **Analytic generator.**
It is understood that the group $(\tau_t)_{t\in\mathbb{R}}$ is pointwise continuous: for any $a\in A$, $\|\tau_t(a)-a\|\to 0$ when $t\to 0$. Let us recall that the analytical generator $\tau_{i/2}$ of a (pointwise continuous) one parameter group $(\tau_t)_{t\in\mathbb{R}}$ of *-automorphisms of a C^*-algebra A is the linear operator acting on A in the following way:

For any $a,b\in A$: $a\in D(\tau_{i/2})$ and $b=\tau_{i/2}(a)$ if and only if there exists a mapping $z\mapsto a_z\in A$ continuous on the strip $\{z\in\mathbb{C}:\ \Im z\in[0,1/2]\}$ and holomorphic in the interior of this strip such that $a_t=\tau_t(a)$ for all $t\in\mathbb{R}$ and $a_{i/2}=b$.

It is known that $\tau_{i/2}$ is a closed linear mapping, $D(\tau_{i/2})$ is a dense subalgebra and $\tau_{i/2}$ is multiplicative. Moreover $\tau_{i/2}(a)^*\in D(\tau_{i/2})$ and $\tau_{i/2}(\tau_{i/2}(a)^*)^*=a$ for any $a\in D(\tau_{i/2})$.

34.6. **Second Pentagon Relation** $\widetilde{W}^*=\widetilde{W}^*\#W$**.**

Theorem 34.1 *We have the second pentagon relation as following:*

$$\widetilde{W}_{13}\widetilde{W}_{12}W_{23}=W_{23}\widetilde{W}_{12}$$

Proof.

$$
\begin{aligned}
&(\overline{z}\otimes u\otimes r|LHS|\overline{x}\otimes y\otimes s)\\
&= \sum_{k,m,n}(\overline{z}\otimes r|\widetilde{W}|\overline{e}_m\otimes e_n)(\overline{e}_m\otimes u|\widetilde{W}|\overline{x}\otimes e_k)(e_k\otimes e_n|W|y\otimes s)\\
&= \sum_{k,m,n}(\hat{Q}x\otimes u|W|\hat{Q}^{-1}e_m\otimes e_k)(\hat{Q}e_m\otimes r|W|\hat{Q}^{-1}z\otimes e_n)(e_k\otimes e_n|W|y\otimes s)\\
&= (\hat{Q}x\otimes u\otimes r|W_{12}W_{13}W_{23}|\hat{Q}^{-1}z\otimes y\otimes s)\\
&= (\hat{Q}x\otimes u\otimes r|W_{23}W_{12}|\hat{Q}^{-1}z\otimes y\otimes s)\\
&= \sum_k(u\otimes r|W|e_k\otimes s)(\hat{Q}x\otimes e_k|W|\hat{Q}^{-1}z\otimes y)\\
&= \sum_k(u\otimes r|W|e_k\otimes s)(\overline{z}\otimes e_k|\widetilde{W}|\overline{x}\otimes y)\\
&= (\overline{z}\otimes u\otimes r|W_{23}\widetilde{W}_{12}|\overline{x}\otimes y\otimes s)\\
&= (\overline{z}\otimes u\otimes r|RHS|\overline{x}\otimes y\otimes s)
\end{aligned}
$$

□

We remark that we have the following:

$$
\begin{aligned}
(x\otimes u|W|z\otimes y) &= (\overline{z}\otimes Qu|\widetilde{W}|\overline{x}\otimes Q^{-1}y)\\
(x\otimes u|W|z\otimes y) &= (\overline{z}\otimes Qu|\widetilde{W}|\overline{x}\otimes Q^{-1}y)\\
(x\otimes u|W|z\otimes y) &= (\overline{z}\otimes Qu|\widetilde{W}|\overline{x}\otimes Q^{-1}y)\\
(x\otimes Q^{-1}u|W|z\otimes Qy) &= (\overline{z}\otimes u|\widetilde{W}|\overline{x}\otimes y)\\
(\hat{Q}x\otimes u|W|\hat{Q}^{-1}z\otimes y) &= (\overline{z}\otimes u|\widetilde{W}|\overline{x}\otimes y)
\end{aligned}
$$

Slices of $\widetilde{W}$ = Slices of $W= A$

Theorem 34.2 *A is a C^*-algebra.*

Proof. We use the second pentagon formula:

$$\widetilde{W}_{12}W_{23}\widetilde{W}^*_{12} = \widetilde{W}^*_{13}W_{23}$$

Since Slice of LHS= A and Slices of RHS= A^*A, we have $A = A^*A = A^*$ □

Lecture 2: Pontryagin duality in quantum group
(12^{th} September 2016)

We shall consider unitary operators $X, Y, Z, \dots$ acting on tensor product of Hilbert spaces.

Definition 34.3 Let H_1, H_2, H_3 be Hilbert spaces and $X \in \mathrm{B}(H_1 \otimes H_2)$ and $Y \in \mathrm{B}(H_2 \otimes H_3)$ be unitary operators. We say that pair (X, Y) is $\#$-composable if operator $X_{12}^* Y_{23} X_{12} Y_{23}^* \in \mathrm{B}(H_1 \otimes H_2 \otimes H_3)$ has trivial second leg. Then $X_{12}^* Y_{23} X_{12} Y_{23}^* = Z_{13}$, where Z is a unitary operator acting on $H_1 \otimes H_3$. In what follows, we shall write

$$Z = X \# Y.$$

For $X \in \mathrm{B}(H_1 \otimes H_2)$ we set

$$\begin{pmatrix} \text{Left slices} \\ \text{of } X \end{pmatrix} = \left[(\mu \otimes \mathrm{id})X : \mu \in \mathrm{B}(H_1)_* \right],$$

$$\begin{pmatrix} \text{Right slices} \\ \text{of } X \end{pmatrix} = \left[(\mathrm{id} \otimes \nu)X : \nu \in \mathrm{B}(H_2)_* \right],$$

Proposition 34.4 Assume that $X \# Y = Z$. Then

$$\begin{pmatrix} \text{Left slices} \\ \text{of } Y \end{pmatrix} = \begin{pmatrix} \text{Left slices} \\ \text{of } Z \end{pmatrix} \begin{pmatrix} \text{Left slices} \\ \text{of } Y \end{pmatrix}$$
$$\begin{pmatrix} \text{Right slices} \\ \text{of } X \end{pmatrix} = \begin{pmatrix} \text{Right slices} \\ \text{of } X \end{pmatrix} \begin{pmatrix} \text{Right slices} \\ \text{of } Z \end{pmatrix}$$

Proof. $X \# Y = Z$ so according to Definition 34.3, $X_{12}^* Y_{23} X_{12} Y_{23}^* = Z_{13}$ so

- $X_{12}^* Y_{23} X_{12} = Z_{13} Y_{23}$ and then slice the first and second leg of this equality then we obtain the first relation.
- $Y_{23} X_{12} Y_{23}^* = X_{12} Z_{13}$ and by slicing the second and third leg og this equality we obtain the second relation.

$\square$

$$A = \begin{pmatrix} \text{Left slices} \\ \text{of } W \end{pmatrix}.$$

$$\widehat{A} = \begin{pmatrix} \text{Right slices} \\ \text{of } W^* \end{pmatrix}.$$

34.7. **Adapted operators.**

Definition 34.5 Let H and K be Hilbert spaces and $W \in \mathrm{B}(H \otimes H)$ and $V \in \mathrm{B}(K \otimes H)$ be unitary operators. We say that V is adapted to W if $(V.W)$ is $\#$-composable and $V \# W = V$. It means that

$$W_{23} V_{12} = V_{12} V_{13} W_{23}.$$

If V is adapted to W and the second leg of V is affiliated to A then

$$(\mathrm{id} \otimes \Delta)V = W_{23} V_{12} W_{23}^* = V_{12} V_{13}.$$

Let H and K be separable Hilbert spaces, $W \in \mathrm{B}(H \otimes H)$ be a manageable multiplicative unitary, $V \in \mathrm{B}(K \otimes H)$ be a unitary adapted to W and

$$B = \begin{pmatrix} \text{Right slices} \\ \text{of } V^* \end{pmatrix}.$$

Then, using the notation introduced in the previous theorem we have:

0. There exists a unitary $\widetilde{V} \in \mathrm{B}(\overline{K} \otimes H)$ such that

$$(r \otimes u | V | s \otimes y) = \left(\overline{s} \otimes Qu \middle| \widetilde{V} \middle| \overline{r} \otimes Q^{-1} y \right)$$

for any $r, s \in K$, $y \in D(Q^{-1})$ and $u \in D(Q)$. One could say that V is 'semi-manageable'.

$$\begin{pmatrix} \text{Right slices} \\ \text{of } V^* \end{pmatrix} = \begin{pmatrix} \text{Right slices} \\ \text{of } \widetilde{V}^{*\,\top\otimes\top} \end{pmatrix}$$

1. B is a separable C^*-algebra acting on K in a non-degenerate way.

2. $V \in \mathrm{M}(B \otimes A)$.

3. $(\mathrm{id} \otimes \Delta)V = V_{12}V_{13}$.

4. For any $\varphi \in \mathrm{B}(K)_*$ we have: $(\varphi \otimes \mathrm{id})V \in D(\tilde{\kappa})$ and

$$\tilde{\kappa}\Big((\varphi \otimes \mathrm{id})V\Big) = (\varphi \otimes \mathrm{id})(V^*).$$

5. $V^{\top\otimes R} = \widetilde{V}^*$.

6. The C^*-algebra B is generated by $V \in \mathrm{M}(B \otimes A)$ in the following sense: for any Hilbert space L, any $\pi \in \mathrm{Rep}(B, L)$ and any non-degenerate separable C^*-algebra $D \subset \mathrm{B}(L)$ we have:

$$\Big((\pi\otimes\mathrm{id})V \in \mathrm{M}(D\otimes A)\Big) \Longrightarrow \Big(\pi \in \mathrm{Mor}(B, D)\Big)$$

34.8. **Derived pentagon equations.**

Now H is a separable Hilbert space, $W \in \mathrm{B}(H \otimes H)$ is a manageable unitary and Q and $\widetilde{W}$ are the operators related to W.

Proposition 34.6 Let K be a Hilbert space and $V \in \mathrm{B}(K \otimes H)$ be a unitary operator adapted to W. Then

1. $\left(V^{*\,\top\otimes\top}, \widetilde{W}\right)$ is #-composable and operators V and $\widetilde{V} = V^{*\,\top\otimes\top} \# \widetilde{W}$ is the operator showing the semi-manageability of V

2. $\left(\widetilde{V}^*, W\right)$ is #-composable and $\widetilde{V}^* \# W = \widetilde{V}^*$.

3. $\left(\widetilde{V}^{\top\otimes\top}, \widetilde{W}\right)$ is #-composable and $\widetilde{V}^{\top\otimes\top} \# \widetilde{W} = V^*$.

4. Then there exists a unitary operator $Z \in \mathrm{B}(\overline{H} \otimes H)$ (depending on W only) such that $Z^{\top\otimes\top} = \Sigma Z \Sigma$ and

$$\widetilde{V}_{13} = Z_{23} V_{12}^{*\,\top\otimes\top} Z_{23}^*$$

for any unitary operator V adapted to W.

Proof. Let

$$Z = \Sigma \widetilde{W}^{\top\otimes\top} \Sigma \widetilde{W}.$$

Then by easy computation $Z^{\top\otimes\top} = \Sigma Z \Sigma$. Let V be a unitary adapted to W. Then $\widetilde{V} = V^{*\,\top\otimes\top} \# \widetilde{W}$ and $\widetilde{V}^{\top\otimes\top} \# \widetilde{W} = V^*$. Expanding these relations we obtain

$$\widetilde{W}_{23} V_{12}^{*\,\top\otimes\top} \widetilde{W}_{23}^* = V_{12}^{*\,\top\otimes\top} \widetilde{V}_{13},$$

$$\widetilde{W}_{23}^{\top\otimes\top} V_{13}^{*\,\top\otimes\top} \widetilde{V}_{12} \widetilde{W}_{23}^{*\,\top\otimes\top} = \widetilde{V}_{12}$$

and the formula follows. □

Application of Proposition 34.6 1. :

$$V^{*\,\top\otimes\top} \# \widetilde{W} = \widetilde{V},$$

$$V \# \widetilde{W}^{*\,\top\otimes\top} = \widetilde{V}^{*\,\top\otimes\top},$$

$$\begin{pmatrix} \text{Right slices} \\ \text{of } V \end{pmatrix} = \begin{pmatrix} \text{Right slices} \\ \text{of } V \end{pmatrix} \begin{pmatrix} \text{Right slices} \\ \text{of } \widetilde{V}^{*\,\top\otimes\top} \end{pmatrix},$$

$$B^* = B^* B,$$

B is a C*-algebra.
Proposition 34.6 is valid for every operator which adapted to W and it is straightforward that a unitary operator is adapted to itself iff it be a multiplicative unitary so if we replace V by multiplicative unitaries W and W^* we conclude that:

- A is a C*-algebra.
- $\widehat{A}$ is a C*-algebra.

Application of Proposition 34.6 4. : For $V = W$ we have:

$$\widetilde{W}_{13} = Z_{23} {W^*_{12}}^{\top\otimes\top} Z^*_{23}$$

$$A = \begin{pmatrix} \text{Left} & \text{slices} \\ \text{of } \widetilde{W} \end{pmatrix}, \qquad A^\top = \begin{pmatrix} \text{Left} & \text{slices} \\ \text{of } {W^*_{12}}^{\top\otimes\top} \end{pmatrix}.$$

Therefore there exists an antiisomorphism of C*-algebras

$$A \ni a \longrightarrow a^R \in A$$

such that

$$I \otimes a^R = Z(a^\top \otimes I)Z^*.$$

This is the unitary coinverse (unitary antipod)

34.9. **Pontryagin duality in quantum groups.** In classical case, for a locally compact abelian group G we can define the dual group $\widehat{G}$ which is the set of all characters on G equipped with uniform convergence on compact sets topology. The Pontryagin duality theorem states that a locally compact abelian group G identify naturally with their bidual $\widehat{\widehat{G}}$. But in abelian case, every character of G is an irreducible representation and vice versa. Therefore a natural question arises about representations in quantum versions. The answer is adapted operators. In this section we try to construct the dual quantum group by using the adapted operators.

Definition 34.1 ${}^uW$ is called the universal operator adapted to W if for every operator V adapted to W there exists a unique $\pi \in \mathrm{Mor}(\widehat{A}^u, B)$ such that $(\pi \otimes \mathrm{id}){}^uW = V$, where $\widehat{A}^u$ and B are the C*-algebras of right slices of W and V respectively.

$${}^uW \# W = {}^uW$$
$${}^uW \in \mathrm{M}(\widehat{A}^u \otimes A)$$
$$\widehat{\Delta}^u \in \mathrm{Mor}(\widehat{A}^u \otimes \widehat{A}^u)$$
$$(\widehat{\Delta}^u \otimes \mathrm{id})\,{}^uW = {}^uW_{23}\,{}^uW_{13}.$$

For every locally compact quantum group, the universal adapted operator to W exists(It is the direct sum of adapted unitaries). But for non-amenable quantum groups the universal C*-algebra $\widehat{A}^u$ is bigger than $\widehat{A}$, i.e. there exists a morphism $\widehat{\pi} : \widehat{A}^u \to \widehat{A}$ such that $(\widehat{\pi} \otimes \mathrm{id}){}^uW = W$(A quantum group is amenable if and only if $\ker \widehat{\pi}$ is trivial).

By replacing W by $\widehat{W}$, we find W^u, π and A^u such that:

$$(\mathrm{id} \otimes \pi)W^u = W$$
$$W^u \in \mathrm{M}(A \otimes A^u)$$
$$\pi : A^u \to A$$

$$\begin{tikzcd} \widehat{A}^u \otimes A^u \arrow[r] \arrow[d] & \widehat{A}^u \otimes A \arrow[d] \\ \widehat{A} \otimes A^u \arrow[r] & \widehat{A} \otimes A \end{tikzcd}$$

Now we put $\overset{u}{W} = {}^{u}W \# W^{u}$ then $\overset{u}{W} \in \widehat{A}^{u} \otimes A^{u}$ and $(\widehat{\pi} \otimes \pi)\, \overset{u}{W} = W$

$$\begin{tikzcd} \overset{u}{W} \arrow[r, mapsto] \arrow[d, mapsto] & {}^{u}W \arrow[d, mapsto] \\ W^{u} \arrow[r, mapsto] & W \end{tikzcd}$$

Lecture 3: Quantum group of all characters
(14^{th} September 2016)

35. Introduction

In this section we concern the anti-isomorhism between two categories. one category is locally compact topological spaces and the second category is the category of commutative C*-algebras.

Category	Objects	Morphisms
LC topological spaces	LC topological spaces	continuous maps
Commutative C^*-algebras	Commutative C^*-algebras	?

What are the morphisms in the second category? The correspondence between object is easy. For a locally compact space Λ, $C_0(\Lambda)$, continuous functions on Λ which vanish at infinity, is a commutative C*-algebra correspondence to Λ. Now Take a continuous map $\phi : \Lambda' \to \Lambda$. We guess that the morphism between $C_0(\Lambda)$ and $C_0(\Lambda')$ must be defined by

$$\phi_* : C_0(\Lambda) \to C_0(\Lambda) \qquad \phi_*(a)(\lambda') = a(\phi(\lambda'))$$

for every $a \in C_0(\Lambda)$ and $\lambda' \in \Lambda'$.It is a natural definition and this map is $*$-algebra homomorphism but it is only a bounded continuous map on the other word $\phi_*(C_0(\Lambda)) \subseteq C_b(\Lambda)') = M(C_0(\Lambda'))$ but $\phi_*(C_0(\Lambda)) \nsubseteq C_0(\Lambda')$.

So we can define the morphism between two commutative C*-algebraas A and B follows:

$$\mathrm{Mor}(A, B) = \{\phi : A \to \mathrm{M}(B) : \phi(A)B = B\}$$

which we mean by $\phi(A)B$ the norm closed linear span of the set $\{\phi(a)b : a \in A, b \in B\}$. But why we add the condition $\phi(A)B = B$. The answer is related to decomposition of morphisms because this condition allows us to extend ϕ uniquely $\tilde{\phi} : \mathrm{M}(A) \to \mathrm{M}(B)$. Actually $\phi \circ \psi := \tilde{\phi} \circ \psi$

For instance, Let Λ' be an open subset of Λ and $\phi : \Lambda' \to \Lambda$ be the injection. We have also $\Lambda \setminus \Lambda'$ as a closed subset of Λ. Then $C_0(\Lambda') \hookrightarrow C_0(\Lambda)$ is not a morphism because that condition is not satisfied(we can not get all the algebra $C_0(\Lambda)$ because $C_0(\Lambda')$ is an ideal in $C_0(\Lambda)$)

As we know we can define tensor product of C*-algebras which is corresponded to Cartesian product of topological spaces.Let $f_\lambda : Y \to X$ be a continuous map for $\lambda \in \Lambda$. Then we can define $\phi_\lambda \in \mathrm{Mor}(A, B)$.

Or equivalently we can assume $f : \Lambda \times Y \to X$. Take A, B and Z the commutative C^*-algebras corresponding to Y, X and Λ respectively. So there exists $\phi \in \mathrm{Mor}(A, Z \otimes B)$ corresponded to f. In special case, take the quantum space of all maps from Y to X, X^Y, as Λ and D the C*-algebra corresponded to X^Y then the map $X^y \times Y \to X$ has a universal propert which says:
"For every $\phi \in \mathrm{Mor}(A, C \otimes B)$ there exists a unique $\psi \in \mathrm{Mor}(D, C)$ such that $\phi = (\psi \otimes \mathrm{id})\Phi$ for some $\Phi \in \mathrm{Mor}(A, D \otimes B)$.

Theorem 35.1 *If Y be a finite set (means $\dim B < \infty$) and A is unital then there exists a universal $(D, \Phi \in \mathrm{Mor}(A, D \otimes B))$.*

Quantum group is a quantum space G with a associative bilinear map $G \times G \to G$ then the C^*-algebra A and a coassociative morphism $\Delta \in \mathrm{Mor}(A, A \otimes A)$ are corresponded to the quantum group. We say $\chi : G \to S^1$ such that $\chi(gg') = \chi(g)\chi(g')$ are characters of G. Since continuous functions with value in S^1 are corresponded to unitary elements of $\mathrm{M}(A)$ so we nedd unitaries $\tilde{\chi} \in \mathrm{M}(A)$ such that $\Delta(\tilde{\chi}) = \tilde{\chi} \otimes \tilde{\chi}$. But we have such a these unitaries less than we need so we are going to describe them as follows:

$$\chi : \Lambda \times G \to S^1 \qquad \chi(\lambda, g)\chi(\lambda, g')$$

so we need $V \in \mathrm{M}(Z \otimes A)$ such that $(\mathrm{id} \otimes \Delta)V = V_{23}V_{13}$. In classical case these are exactly the representations of group.
Now the quantum group of all characters must be ${}^uW \in \mathrm{M}(\widehat{A}^u \otimes A)$ such that $(\mathrm{id} \otimes \Delta){}^uW = {}^uW^u_{12}W_{13}$.
${}^uW$ should be universal, i.e. for every $V \in \mathrm{M}(C \otimes A)$ such that $(\mathrm{id} \otimes \Delta)V = V_{12}V_{13}$ there exists a unique $\alpha \in \mathrm{Mor}(\widehat{A}^u, C)$ such that $V = (\alpha \otimes \mathrm{id}){}^uW$.

Lecture 4: Multiplicative unitary
(15^{th} September 2016)

The theory of multiplicative unitary operators has played a central role in the modern approach to quantum groups. A unitary operator $W \in B(H, H)$ is called multiplicative if it satisfies the pentagon equation

$$W_{23}W_{12} = W_{12}W_{13}W_{23}$$

However this condition alone does not guarantee that W is a multiplicative unitary related to a quantum group.
For the purely algebraic context of this lecture we refer to the first lecture of A. Van Daele in this lecture note.

Let $U \in \mathrm{M}(C \otimes A)$ and $U' \in \mathrm{M}(C' \otimes A)$ be unitary operators. Then we will denote $(\mathrm{id} \otimes \mu)U$ by $U_{1\mu} \in \mathrm{M}(C)$ for every $\mu \in A_*$(Note that A_* is well defined if we see A as a subalgebra of $B(H)$). Suppose that there exists a morphism $\beta \in \mathrm{Mor}(C', C)$ such that

$$U = (\beta \otimes \mathrm{id})U'$$

Then $\beta(U'_{1\mu}) = U_{1\mu}$. It is easy to conclude that

$$\| U_{1\mu} \| \leq \| U'_{1\mu} \| \tag{35.1}$$

On the other hand (35.1) is a sufficient condition to obtain $\beta \in \mathrm{Mor}(C', C)$

We will have a universal object ${}^uW$ if for every $\mu \in A_*$,

$$\|{}^uW_{1\mu} \| = \sup_U \| U_\mu \|$$

36. Homomorphism of quantum groups

Definition 36.1 Let W and W' be multiplicative unitaries regarding to quantum groups (A, Δ_A) and (B, Δ_B) respectively. Then a homomorphism between a and B is a morphism $\alpha \in \mathrm{Mor}(A^u, B)$ which satisfies the following condition:

$$\Delta_B(\alpha(x)) = (\alpha \otimes \alpha)\Delta^u_A(x)$$

If such a morphism exists, then for the unitary $W^u \in \mathrm{M}(A^u \otimes A^u)$, constructed in the second lecture,

$$(\mathrm{id} \otimes \alpha)W^u = \overset{u}{V} \in \mathrm{M}(\widehat{A} \otimes B)$$

is a bicharacter. Which means that

$$W \# \overset{u}{V} = \overset{u}{V}$$
$$\overset{u}{V} \# W' = \overset{u}{V}$$

37. Heisenberg and anti-Heisenberg pairs

Let (A, Δ_A) be the quantum group generated by the manageable unitary V.

$$(\mathrm{id} \otimes \Delta_A)V = V_{12}V_{13}$$
$$(\widehat{\Delta}_A \otimes \mathrm{id})V = V_{23}V_{13}$$

Actually V is just W but forget which Hilbert space it acts. By construction, we have inclusion maps $\pi : A \to B(H)$ and $\widehat{\pi} : \widehat{A} \to B(H)$, which are faithful $*$-homomorphisms (In fact they are embedding). Then we have

$$W\#W = W = (\widehat{\pi}\otimes\pi)V$$
$$(\widehat{\pi}\otimes\pi)V\#(\widehat{\pi}\otimes\pi)V = (\widehat{\pi}\otimes\pi)V$$
$$(\widehat{\pi}\otimes\pi)((\mathrm{id}\otimes\pi)V\#(\widehat{\pi}\otimes\mathrm{id})V) = (\widehat{\pi}\otimes\pi)V$$

Sowe conclude that

$$(\mathrm{id}\otimes\pi)V\#(\widehat{\pi}\otimes\mathrm{id})V = V$$

We can construct Heisenberg pair in universal version of quantum group but we must note that in this case the Heisenberg pairs are not necessarily faithful. ($\ker\widehat{\pi}^u = \ker\widehat{\Lambda}$ where $\widehat{\Lambda}:\widehat{A}^u\to\widehat{A}$)

Theorem 37.1 *"Everything" is determined by* (A,Δ_A).

Everything includes:

- ultra weak topology on A and $\widehat{A}$ (By existence of Heisenberg pair)
- co-inverse, unitary co-inverse, scaling group.
- $(\widehat{A},\widehat{\Delta})$ and $V\in \mathrm{M}(\widehat{A}\otimes A)$
- everything exclude W.

We will denote $(\mathrm{id}\otimes\pi)V$ and $(\widehat{\pi}\otimes\mathrm{id})V$ by $V_{1\pi}$ and $V_{\widehat{\pi}3}$ respectively. Then

$$V_{\widehat{\pi}3}V_{1\pi} = V_{1\pi}V_{13}V_{\widehat{\pi}3}$$
$$V_{1\pi}V_{13} = V_{\widehat{\pi}3}V_{1\pi}V^*_{\widehat{\pi}3}$$

so

$$(\pi\otimes\mathrm{id})\Delta(a) = V_{\widehat{\pi}2}(\pi(a)\otimes 1)V^*_{\widehat{\pi}2} \tag{37.1}$$

A. Van Daele defined another Pentagon equation as follows:

$$V_{12}V_{23} = V_{23}V_{13}V_{12}$$

Then we can define $\rho: A\to B(K)$ and $\widehat{\rho}:\widehat{A}\to B(K)$ such that

$$V_{1\rho}V_{\widehat{\rho}3} = V_{\widehat{\rho}3}V_{13}V_{1\rho}$$

The pair $(\widehat{\rho},\rho)$ is called anit-Heisenberg pair. It can be shown that a faithful anti-Heisenberg pair exists and if $(\pi,\widehat{\pi})$ acts on a Hilbert space H, then $(\widehat{\rho},\rho)$ acts on $\bar{H}$. Moreover

$$\rho(a) = \pi(a^R)^T$$
$$\widehat{\rho}(\widehat{a}) = \widehat{\pi}(\widehat{a}^{\widehat{R}})^T$$

and

$$(\mathrm{id}\otimes\rho)\Delta(a) = \Sigma[V^*_{\widehat{\rho}2}(\rho(a)\otimes 1)V_{\widehat{\rho}2}] \tag{37.2}$$

If we apply ρ to the (37.1) then it will be equal to (37.2) applied by π.

Remark 37.1 ultra weak topology on A does not depend on which Heisenberg pair you get.

Lecture 5: Crossed product
(16^{th} September 2016)

Notation. Let X and Y be a norm closed subsets of a C* algebra. We set

$$XY = \left\{ xy : \begin{array}{c} x \in X \\ y \in Y \end{array} \right\}^{\mathrm{CLS}},$$

where CLS stands for norm Closed Linear Span.

Let C* be a category whose objects are separable C* algebras. If $A, B \in$ C* then by definition $\mathrm{Mor}(A, B)$ is the set of all *-algebra homomorphisms φ acting from A into $M(B)$ such that $\varphi(A)B = B$. Any $\varphi \in \mathrm{Mor}(A, B)$ admits a unique extension to a unital *-algebra homomorphism acting from $M(A)$ into $M(B)$. Composition of morphisms is defined as composition of their extensions.

In what follows

$$\varphi : A \longrightarrow B$$

means that $\varphi \in \mathrm{Mor}(A, B)$. It does not imply that $\varphi(A) \subset B$.

38. The concept of Crossed Product Algebra

Let A, B, C be C*-algebras, $\alpha \in \mathrm{Mor}(A, C)$ and $\beta \in \mathrm{Mor}(B, C)$. We say that C is a crossed product of A and B if

$$\alpha(A)\beta(B) = C.$$

Example 38.1 Let A and B be C^*-algebras. Then $C = A \otimes B$ is a crossed product of A and B with repect to the following morphisms.

$$\alpha(a) = a \otimes I_B \quad a \in A$$
$$\beta(b) = I_A \otimes b \quad b \in B$$

38.1. **Crossed Product Algebra in practice.** Let A, B be separable C* algebras, H be a Hilbert space, $\alpha \in \mathrm{Rep}(A, H)$ and $\beta \in \mathrm{Rep}(B, H)$. Then

$$\alpha(A)\beta(B) = \beta(B)\alpha(A) \quad \text{if and only if} \quad \alpha(A)\beta(B) \text{ is a C}^* \text{ algebra.}$$

Moreover in this case $\alpha \in \mathrm{Mor}(A, C)$ and $\beta \in \mathrm{Mor}(B, C)$, where $C = \alpha(A)\beta(B)$. Therefore C is a crossed product of A and B.

Locally compact quantum groups appear in dual pairs:

$$G = (A, \Delta)$$

$$\widehat{G} = (\widehat{A}, \widehat{\Delta})$$

The duality is described by a bicharacter V. This is a unitary element of $M(\widehat{A} \otimes A)$ such that

$$(\mathrm{id} \otimes \Delta)V = V_{12}V_{13},$$
$$(\widehat{\Delta} \otimes \mathrm{id})V = V_{23}V_{13}.$$

38.2. **Heisenberg pairs.** Let H be a Hilbert space and

$$\pi \in \mathrm{Rep}(A, H),$$

$$\widehat{\pi} \in \mathrm{Rep}(\widehat{A}, H).$$

We say that the pair $(\pi, \widehat{\pi})$ is a Heisenberg pair acting on H if

$$V_{\widehat{\pi}3}V_{1\pi} = V_{1\pi}V_{13}V_{\widehat{\pi}3}.$$

The existence of faithful Heisenberg pairs is one of the basic features of the theory of locally compact quantum group. In good cases of so called regular groups (including classical locally compact quantum groups) there is only one (up to unitary equivalence and multiplicity) Heisenberg pair. The class of regular quantum groups includes all classical locally compact groups. In general however we may have many nonequivalent Heisenberg pairs.

38.3. **Comultiplication formula I.** Let π and $\widehat{\pi}$ be representations of A and $\widehat{A}$ acting on the same H. Then the following three conditions are equivalent:
$(\pi, \widehat{\pi})$ is a Heisenberg pair.
For any $a \in A$ we have

$$(\pi \otimes \mathrm{id})\Delta(a) = V_{\widehat{\pi}2}(\pi(a) \otimes I)V_{\widehat{\pi}2}^{*}.$$

For any $\widehat{a} \in \widehat{A}$ we have

$$(\widehat{\pi} \otimes \mathrm{id})\widehat{\Delta}(\widehat{a}) = \widehat{V}_{\pi 2}(\widehat{\pi}(\widehat{a}) \otimes I)\widehat{V}_{\pi 2}^{*}.$$

38.4. **Anti-Heisenberg pairs.** Let H be a Hilbert space and

$$\rho \in \mathrm{Rep}(A, H),$$

$$\widehat{\rho} \in \mathrm{Rep}(\widehat{A}, H).$$

We say that the $(\rho, \widehat{\rho})$ is an anti-Heisenberg pair acting on H if

$$V_{1\rho}V_{\widehat{\rho}3} = V_{\widehat{\rho}3}V_{13}V_{1\rho}.$$

Existence of faithful anti-Heisenberg pairs?
Starting from a faithful Heisenberg pair $(\pi, \widehat{\pi})$ acting on H and setting

$$\rho(a) = \pi(a^R)^\top$$
$$\widehat{\rho}(\widehat{a}) = \widehat{\pi}(\widehat{a}^{\widehat{R}})^\top$$

we obtain a faithful anti-Heisenberg pair $(\rho, \widehat{\rho})$ acting on $\overline{H}$.

38.5. **Comultiplication formula II.** Let ρ and $\widehat{\rho}$ be representations of A and $\widehat{A}$ acting on the same K. Then the following three conditions are equivalent:
$(\rho, \widehat{\rho})$ is an anti-Heisenberg pair.
For any $a \in A$ we have

$$(\mathrm{id} \otimes \rho)\Delta(a) = \widehat{V}_{1\widehat{\rho}}(I \otimes \rho(a))\widehat{V}_{1\widehat{\rho}}^{*}.$$

For any $\widehat{a} \in \widehat{A}$ we have

$$(\mathrm{id} \otimes \widehat{\rho})\widehat{\Delta}(\widehat{a}) = V_{1\rho}(I \otimes \widehat{\rho}(\widehat{a}))V_{1\rho}^{*}.$$

38.6. **Heisenberg versus anti-Heisenberg I.**

Theorem 38.1 Let $(\pi, \widehat{\pi})$ be a Heisenberg pair acting on a Hilbert space H and $(\rho, \widehat{\rho})$ be an anti-Heisenberg pair acting on a Hilbert space K. Then $\pi \otimes \widehat{\rho}$ and $\rho \otimes \widehat{\pi}$ are unitarily equivalent representations of $A \otimes \widehat{A}$ acting on $H \otimes K$ and $K \otimes H$ respectively.More precisely, unitary operator

$$V_{\widehat{\rho}\pi}\Sigma V_{\widehat{\pi}\rho} : H \otimes K \longrightarrow K \otimes H$$

intertwines $\pi \otimes \widehat{\rho}$ with $\rho \otimes \widehat{\pi}$.

π is quasi-equivalent to ρ and $\widehat{\pi}$ is quasi-equivalent to $\widehat{\rho}$.

There is a unique quasi-equivalence class of representations of A that contains first elements of all Heisenberg and anti-Heisenberg pairs. Similarly there is a unique quasi-equivalence class of representations of $\widehat{A}$ that contains second element of all Heisenberg and anti-Heisenberg pairs.

- Algebras A and $\widehat{A}$ are endowed with natural ultra weak topologies.
- Representations π, $\widehat{\pi}$, ρ, $\widehat{\rho}$ are faithful.

We say that a linear functional on A is normal if it is continuous with respect to the ultra-weak topology. Let A_* be the set of all normal functionals on A. Similarly one defines the set $\widehat{A}_*$ of all normal functionals on $\widehat{A}$. Then we have:

$$A = \left\{(\widehat{\omega} \otimes \mathrm{id})V : \widehat{\omega} \in \widehat{A}_*\right\}^{\text{norm closure}}$$

$$\widehat{A} = \left\{(\mathrm{id} \otimes \omega)V : \omega \in A_*\right\}^{\text{norm closure}}$$

So, slicing V with the normal functionals we obtain dense subsets of A and $\widehat{A}$.

38.7. **Heisenberg versus anti-Heisenberg II.**

Theorem 38.2 Let H, K be Hilbert spaces and

$$\begin{array}{ll} \pi \in \mathrm{Rep}(A, H) & \qquad \rho \in \mathrm{Rep}(A, K) \\ \widehat{\pi} \in \mathrm{Rep}(\widehat{A}, H) & \qquad \widehat{\rho} \in \mathrm{Rep}(\widehat{A}, K) \end{array}$$

Then any two of the three conditions below imply the third one.

(1) $(\pi, \widehat{\pi})$ is a Heisenberg pair acting on H

(2) $(\rho, \widehat{\rho})$ is an anti-Heisenberg pair acting on K

(3) For any $a \in A$ and $\widehat{a} \in \widehat{A}$, commutator

$$\left[(\pi \otimes \rho)\Delta(a), (\widehat{\pi} \otimes \widehat{\rho})\widehat{\Delta}(\widehat{a})\right] = 0$$

Proof. Condition 1 means that

$$V_{1\pi}^* V_{\widehat{\pi}3} V_{1\pi} V_{\widehat{\pi}3}^* = V_{13}$$

Condition 2 means that

$$V_{\widehat{\rho}3}^* V_{1\rho} V_{\widehat{\rho}3} V_{1\rho}^* = V_{13}$$

Condition 3 means that

$$V_{1\pi} V_{1\rho} \widehat{V}_{2\widehat{\pi}} \widehat{V}_{2\widehat{\rho}} = \widehat{V}_{2\widehat{\pi}} \widehat{V}_{2\widehat{\rho}} V_{1\pi} V_{1\rho}$$

$$V_{1\pi} V_{1\rho} V_{\widehat{\pi}3}^* V_{\widehat{\rho}3}^* = V_{\widehat{\pi}3}^* V_{\widehat{\rho}3}^* V_{1\pi} V_{1\rho}$$

$$V_{\widehat{\rho}3} V_{1\rho}^* V_{\widehat{\pi}3} V_{1\pi}^* = V_{1\rho}^* V_{\widehat{\rho}3} V_{1\pi}^* V_{\widehat{\pi}3}$$

$$V_{\widehat{\rho}3}^* V_{1\rho} V_{\widehat{\rho}3} V_{1\rho}^* = V_{1\pi}^* V_{\widehat{\pi}3} V_{1\pi} V_{\widehat{\pi}3}^*$$

□

38.8. **Stronger version of Condition 3.** We shall use the relation '$\simeq$' (to be explained)

Theorem 38.3 Assume that

(1) $(\pi, \widehat{\pi})$ is a Heisenberg pair and
(2) $(\rho, \widehat{\rho})$ is an anti-Heisenberg pair.

Then

(3)

$$\begin{pmatrix} (\pi \otimes \rho)\Delta(a) \\ (\widehat{\pi} \otimes \widehat{\rho})\widehat{\Delta}(\widehat{a}) \end{pmatrix} \simeq \begin{pmatrix} a \otimes I \\ I \otimes \widehat{a} \end{pmatrix}$$

Proof:

$$\begin{pmatrix} (\pi \otimes \rho)\Delta(a) \\ (\widehat{\pi} \otimes \widehat{\rho})\widehat{\Delta}(\widehat{a}) \end{pmatrix} = \begin{pmatrix} V_{\widehat{\pi}\rho}(\pi(a) \otimes I)V_{\widehat{\pi}\rho}^* \\ V_{\widehat{\pi}\rho}(I \otimes \widehat{\rho}(\widehat{a}))V_{\widehat{\pi}\rho}^* \end{pmatrix} \simeq \begin{pmatrix} \pi(a) \otimes I \\ I \otimes \widehat{\rho}(\widehat{a}) \end{pmatrix}$$

39. Drinfeld double. Commutation relations

Theorem 39.1 Assume that

(1) $(\pi, \widehat{\pi})$ is a Heisenberg pair acting on H and
(2) $(\rho, \widehat{\rho})$ is an anti-Heisenberg pair acting on K.

Let

$$r = (\rho \otimes \pi)\Delta(a)$$
$$s = (\widehat{\rho} \otimes \widehat{\pi})\widehat{\Delta}(\widehat{a})$$

Then r and s are representations of A and $\widehat{A}$ acting on $K \otimes H$ and

$$V_{1r}V_{13}V_{s3} = V_{s3}V_{13}V_{1r}$$

Proof: $V_{1r} = V_{1\rho}V_{1\pi}$ and $V_{s3} = V_{\widehat{\pi}3}V_{\widehat{\rho}3}$. Therefore

$$\begin{aligned} V_{1r}V_{13}V_{s3} &= V_{1\rho}V_{1\pi}V_{13}V_{\widehat{\pi}3}V_{\widehat{\rho}3} = V_{1\rho}V_{\widehat{\pi}3}V_{1\pi}V_{\widehat{\rho}3} \\ &= V_{\widehat{\pi}3}V_{1\rho}V_{\widehat{\rho}3}V_{1\pi} = V_{\widehat{\pi}3}V_{\widehat{\rho}3}V_{13}V_{1\rho}V_{1\pi} = V_{s3}V_{13}V_{1r}. \end{aligned}$$

Remark 39.1

- $(\pi, \widehat{\pi})$ is a Heisenberg pair if and only if
$$V_{1\pi}\widehat{V}_{2\widehat{\pi}} = \widehat{V}_{2\widehat{\pi}}V_{1\pi}V_{12}.$$
- $(\rho, \widehat{\rho})$ is an anti-Heisenberg pair if and only if
$$\widehat{V}_{2\widehat{\rho}}V_{1\rho} = V_{12}V_{1\rho}\widehat{V}_{2\widehat{\rho}}.$$
- $(\pi, \widehat{\pi})$ is a Heisenberg pair for G if and only if $(\widehat{\pi}, \pi)$ is a Heisenberg pair for $\widehat{G}$. Similarly $(\rho, \widehat{\rho})$ is an anti-Heisenberg pair for G if and only if $(\widehat{\rho}, \rho)$ is an anti-Heisenberg pair for $\widehat{G}$.

40. C*-algebras subject to an action of G

Let X be a C*-algebra and $\varphi \in \mathrm{Mor}(X, X \otimes A)$. We say that φ is an action of G on X if

(1)
$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\varphi"'] & X \otimes A \arrow[d, "\varphi\otimes\mathrm{id}"] \\ X \otimes A \arrow[r, "\mathrm{id}\otimes\Delta"'] & X \otimes A \otimes A \end{tikzcd}$$
is a commutative diagram,
(2) $\ker\varphi = \{0\}$,
(3) $\varphi(X)(I \otimes A) = X \otimes A$ (Podle condition).

$$\left(\begin{array}{c}\text{Podle}\\ \text{condition}\end{array}\right) \Longrightarrow \Big(\varphi \in \mathrm{Mor}(X, X \otimes A)\Big)$$

Let us denote the category of all C*-actions on a group G, by C^*_G. Objects are C*-algebras with actions of G. For any $X \in \mathrm{C}^*_G$, the action of G on X will be denoted by φ_X. Morphisms in C^*_G are C*-morphisms intertwining the actions of G:

Let X, Y be C*-algebras with actions of G. We say that a morphism $\gamma \in \mathrm{Mor}(X, Y)$ intertwins the actions of G if the diagram

$$\begin{tikzcd} X \arrow[r, "\varphi_X"] \arrow[d, "\gamma"'] & X \otimes A \arrow[d, "\gamma\otimes\mathrm{id}"] \\ Y \arrow[r, "\varphi_Y"'] & Y \otimes A \end{tikzcd}$$

is commutative.

The set of all such morphisms will be denoted by $\mathrm{Mor}_G(X, Y)$.

Example 40.1

- Any C*-algebra C with the trivial action
$$\varphi_C(c) = c \otimes I \in M(C \otimes A)$$
is an object of C^*_G.
- $A = \mathcal{C}_\infty(G)$ with the action
$$\varphi_A(a) = \Delta(a) \in M(A \otimes A)$$
is an object of C^*_G.
- $C \otimes A$ with the action
$$\varphi_{C\otimes A}(c \otimes a) = c \otimes \Delta(a) \in M((C \otimes A) \otimes A)$$
is an object of C^*_G.
- Let C be a C*-algebra and X be a C*-subalgebra of $M(C \otimes A)$ such that

$$(*) \qquad (\mathrm{id} \otimes \Delta)(X)\,(I \otimes I \otimes A) = X \otimes A.$$

Then X with the action
$$\varphi_X = (\mathrm{id} \otimes \Delta)|_X \in \mathrm{Mor}(X, X \otimes A)$$
is an object of C^*_G.

Any object of C^*_G is (isomorphic to an object) of the above form. Let $X \in \mathrm{C}^*_G$. Denote by $C = X$ the same C*-algebra with the trivial action of G and set $X' = \varphi_X(X)$. Then X' is a C*-subalgebra of $M(C \otimes A)$ satisfying condition $(*)$ and
$$\varphi_X : X \longrightarrow X'$$
is an isomorphism in C^*_G category.

Let us choose a Heisenberg pair $(\widehat{\pi}, \pi)$ acting on a Hilbert space H. $\mathcal{K}(H)$ will denote the algebra of all compact operators acting on H.

Theorem 40.1 Let $X \in \mathrm{C}^*_G$ with the action $\varphi_X \in \mathrm{Mor}(X, X \otimes A)$ and $Y \in \mathrm{C}^*_{\widehat{G}}$ with the action $\widehat{\varphi}_Y \in \mathrm{Mor}(Y, Y \otimes \widehat{A})$. For any $x \in X$ and $y \in Y$ we set
$$\alpha_{XY}(x) = \Big[(\mathrm{id} \otimes \pi)\varphi_X(x)\Big]_{13} \in M(X \otimes Y \otimes \mathcal{K}(H)),$$
$$\beta_{XY}(y) = \Big[(\mathrm{id} \otimes \widehat{\pi})\widehat{\varphi}_Y(y)\Big]_{23} \in M(X \otimes Y \otimes \mathcal{K}(H)).$$
Then
$$\alpha_{XY}(X)\beta_{XY}(Y) = \beta_{XY}(Y)\alpha_{XY}(X).$$

Let $\rho \in \mathrm{Rep}(A, K)$ and $\widehat{\rho} \in \mathrm{Rep}(\widehat{A}, K)$. Then for any $x \in X$ and $y \in Y$ we have
$$\begin{aligned}
(\alpha_{XY} \otimes \rho)\varphi_X(x) &= \Big[((\mathrm{id} \otimes \pi)\varphi_X \otimes \rho)\varphi_X(x)\Big]_{134} \\
&= \Big[(\mathrm{id} \otimes \pi \otimes \rho)(\varphi_X \otimes \mathrm{id})\varphi_X(x)\Big]_{134} \\
&= \Big[(\mathrm{id} \otimes \pi \otimes \rho)(\mathrm{id} \otimes \Delta)\varphi_X(x)\Big]_{134} \\
&= \Big[(\mathrm{id} \otimes (\pi \otimes \rho)\Delta)\varphi_X(x)\Big]_{134} \\
(\beta_{XY} \otimes \widehat{\rho})\widehat{\varphi}_Y(y) &= \Big[((\mathrm{id} \otimes \widehat{\pi})\widehat{\varphi}_Y \otimes \widehat{\rho})\widehat{\varphi}_Y(y)\Big]_{234} \\
&= \Big[(\mathrm{id} \otimes \widehat{\pi} \otimes \widehat{\rho})(\widehat{\varphi}_Y \otimes \mathrm{id})\widehat{\varphi}_Y(y)\Big]_{234} \\
&= \Big[(\mathrm{id} \otimes \widehat{\pi} \otimes \widehat{\rho})(\mathrm{id} \otimes \widehat{\Delta})\widehat{\varphi}_Y(y)\Big]_{234} \\
&= \Big[(\mathrm{id} \otimes (\widehat{\pi} \otimes \widehat{\rho})\widehat{\Delta})\widehat{\varphi}_Y(y)\Big]_{234}
\end{aligned}$$

41. Commutation relations for α_{XY} and β_{XY}.

Let $(\rho, \widehat{\rho})$ be an anti-Heisenberg pair. Then for any $x \in X$ and $y \in Y$ commutator

$$\Big[(\alpha_{XY} \otimes \rho)\varphi_X(x), (\beta_{XY} \otimes \widehat{\rho})\widehat{\varphi}_Y(y)\Big] = 0$$

Therefore

$$\begin{aligned}&(\alpha_{XY} \otimes \rho)\varphi_X(X)(\beta_{XY} \otimes \widehat{\rho})\widehat{\varphi}_Y(Y)\\ &\qquad = (\beta_{XY} \otimes \widehat{\rho})\widehat{\varphi}_Y(Y)(\alpha_{XY} \otimes \rho)\varphi_X(X)\end{aligned}$$

Let K be the carrier Hilbert space of $(\rho, \widehat{\rho})$ and $\mathcal{K}(K)$ be the algebra of all compact operators acting on K. Then

$$\begin{aligned}\varphi_X(X)(I \otimes A) &= X \otimes A,\\ (\mathrm{id} \otimes \rho)\varphi_X(X)\,(I \otimes \rho(A)) &= X \otimes \rho(A),\\ (\mathrm{id} \otimes \rho)\varphi_X(X)\,(I \otimes \mathcal{K}(K)) &= X \otimes \mathcal{K}(K),\\ (\alpha_{XY} \otimes \rho)\varphi_X(X)\,(I \otimes \mathcal{K}(K)) &= \alpha_{XY}(X) \otimes \mathcal{K}(K),\end{aligned}$$

$$\begin{aligned}\widehat{\varphi}_Y(Y)(I \otimes A) &= Y \otimes A,\\ (\mathrm{id} \otimes \widehat{\rho})\widehat{\varphi}_Y(Y)\,(I \otimes \widehat{\rho}(A)) &= Y \otimes \widehat{\rho}(A),\\ (\mathrm{id} \otimes \widehat{\rho})\widehat{\varphi}_Y(Y)\,(I \otimes \mathcal{K}(K)) &= Y \otimes \mathcal{K}(K),\\ (\beta_{XY} \otimes \widehat{\rho})\widehat{\varphi}_Y(Y)\,(I \otimes \mathcal{K}(K)) &= \beta_{XY}(Y) \otimes \mathcal{K}(K),\end{aligned}$$

Multiplying the last formula of the previous slide by $I \otimes \mathcal{K}(K)$ from the right we get

$$\alpha_{XY}(X)\beta_{XY}(Y) \otimes \mathcal{K}(K) = \beta_{XY}(Y)\alpha_{XY}(X) \otimes \mathcal{K}(K).$$

Let

$$X \boxtimes Y = \alpha_{XY}(X)\beta_{XY}(Y).$$

Then $X \boxtimes Y$ is a C*-algebra and

$$\alpha_{XY} \in \mathrm{Mor}(X, X \boxtimes Y),$$

$$\beta_{XY} \in \mathrm{Mor}(Y, X \boxtimes Y).$$

It turns out that the crossed product is independent of the particular choice of Heisenberg pair $(\pi, \widehat{\pi})$.

More precisely if $X \boxtimes' Y$, α'_{XY} and β'_{XY} are the C*-algebra and morphisms constructed with the another choice of (α, β) then there exists unique isomorphism $\phi \in \mathrm{Mor}(X \boxtimes Y, X \boxtimes' Y)$ such that the diagram

$$\begin{tikzcd} & X \arrow[dl, "\alpha_{XY}"'] \arrow[dr, "\alpha'_{XY}"] & \\ X \boxtimes Y \arrow[rr, "\phi"] & & X \boxtimes' Y \\ & Y \arrow[ul, "\beta_{XY}"] \arrow[ur, "\beta'_{XY}"'] & \end{tikzcd}$$

is commutative.

Theorem 41.1 *Let* $X, X' \in \mathrm{C}^*_G$, $r \in \mathrm{Mor}_G(X, X')$ *and* $Y, Y' \in \mathrm{C}^*_G$, $s \in \mathrm{Mor}_G(Y, Y')$. *Then there exists unique* $r \boxtimes s \in \mathrm{Mor}(X \boxtimes Y, X' \boxtimes Y')$ *such that the diagrams*

$$\begin{tikzcd} X \arrow[r, "\alpha_{XY}"] \arrow[d, "r"'] & X \boxtimes Y \arrow[d, "r \boxtimes s"] \\ X' \arrow[r, "\alpha_{X'Y'}"'] & X' \boxtimes Y' \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "\beta_{XY}"] \arrow[d, "s"'] & X \boxtimes Y \arrow[d, "r \boxtimes s"] \\ Y' \arrow[r, "\beta_{X'Y'}"'] & X' \boxtimes Y' \end{tikzcd}$$

are commutative.

If $\ker r = \{0\}$ *and* $\ker s = \{0\}$ *then* $\ker(r \boxtimes s) = \{0\}$.

Proof. Note that $X \boxtimes Y \subset M(X \otimes Y \otimes \mathcal{K}(H))$ and identify $r \boxtimes s$ with $r \otimes s \otimes \mathrm{id}$ restricted to $X \boxtimes Y$. $\square$

42. Crossed product as functor from $\mathrm{C}^*_G \times \mathrm{C}^*_{\widehat{G}}$ into C^*.

The previous theorem allows us to define crossed product $X \boxtimes Y$ for any $X \in \mathrm{C}^*_G$ and $Y \in \mathrm{C}^*_{\widehat{G}}$. We also have crossed product of morphisms.

$\boxtimes$ is a covariant functor acting from the category $\mathrm{C}^*_G \times \mathrm{C}^*_{\widehat{G}}$ into C^*. One can also view projections Proj_1 and Proj_2 as covariant functors acting from the $\mathrm{C}^*_G \times \mathrm{C}^*_{\widehat{G}}$ into C^*. In this language, α and β become natural mappings from Proj_1 and Proj_2 into $\boxtimes$.

If one of the considered algebras is endowed with the trivial action of the group then $\boxtimes$ (for objects and morphisms) reduces to $\otimes$. It turns out that $(\boxtimes, \alpha, \beta)$ with this property is uniquely determined by $A \boxtimes \widehat{A}$.

Let $(\pi, \widehat{\pi})$ be a Heisenberg pair. One can show that $C = \pi(A)\widehat{\pi}(A)$ is a C^*-algebra. In general $C \neq A \boxtimes \widehat{A}$. It depends on the choice of $(\pi, \widehat{\pi})$. However we have

Theorem 42.1 The following three conditions are equivalent:

(1) $C = A \boxtimes \widehat{A}$
(2) There exists faithful $\Psi \in \mathrm{Mor}(C, A \otimes C)$ such that

$$\begin{aligned} \Psi(\pi(a)) &= (\mathrm{id} \otimes \pi)\Delta(a) \\ \Psi(\widehat{\pi}(\widehat{a})) &= I \otimes \widehat{\pi}(\widehat{a}) \end{aligned}$$

(3) There exists faithful $\widehat{\Psi} \in \mathrm{Mor}(C, \widehat{A} \otimes C)$ such that

$$\begin{aligned} \widehat{\Psi}(\pi(a)) &= I \otimes \pi(a) \\ \widehat{\Psi}(\widehat{\pi}(\widehat{a})) &= (\mathrm{id} \otimes \widehat{\pi})\widehat{\Delta}(\widehat{a}) \end{aligned}$$

Proof. $1 \Longrightarrow (2\ \&\ 3)$ The C^*-algebra A with the trivial action of G will be denoted by A_{tr}. Clearly Δ intertwines actions of G on A and $A_{\mathrm{tr}} \otimes A$:

$$\Delta \in \mathrm{Mor}_G(A, A_{\mathrm{tr}} \otimes A).$$

Let $C = A \boxtimes \widehat{A}$. Then $(A_{\mathrm{tr}} \otimes A) \boxtimes \widehat{A} = A_{\mathrm{tr}} \otimes (A \boxtimes \widehat{A}) = A_{\mathrm{tr}} \otimes C$.
One can easily show that $\Psi = \Delta \boxtimes \mathrm{id}$ is a faithful morphism from $A \boxtimes \widehat{A} = C$ into $(A_{\mathrm{tr}} \otimes A) \boxtimes \widehat{A} = A_{\mathrm{tr}} \otimes C$ satisfying the formulae appearing in Condition 2.

Similarly $A \boxtimes (\widehat{A}_{\mathrm{tr}} \otimes \widehat{A}) \simeq \widehat{A}_{\mathrm{tr}} \otimes (A \boxtimes \widehat{A}) = \widehat{A}_{\mathrm{tr}} \otimes C$ and $\widehat{\Psi} = \mathrm{id} \boxtimes \widehat{\Delta}$ is a faithful morphism from C into $\widehat{A} \otimes C$ satisfying the formulae appearing in Condition 3. $\square$

Let $(\pi, \widehat{\pi})$ be a Heisenberg pair. Then

$$\begin{array}{ccc} \begin{pmatrix} \pi(a) \\ \widehat{\pi}(\widehat{a}) \end{pmatrix} & & \begin{pmatrix} (\mathrm{id} \otimes \pi)\Delta(a) \\ I \otimes \widehat{\pi}(\widehat{a}) \end{pmatrix} \\ & & \wr\!\wr \\ \begin{pmatrix} I \otimes \pi(a) \\ (\mathrm{id} \otimes \widehat{\pi})\widehat{\Delta}(\widehat{a}) \end{pmatrix} & \simeq & \begin{pmatrix} \Big[(\mathrm{id} \otimes \alpha)\Delta(a)\Big]_{13} \\ \Big[(\mathrm{id} \otimes \widehat{\pi})\widehat{\Delta}(\widehat{a})\Big]_{23} \end{pmatrix} \end{array}$$

43. Universal $A \boxtimes \widehat{A}$

Let $(\pi,\widehat{\pi})$ be a universal R-Heisenberg pair and $C = \pi(A)\widehat{\pi}(\widehat{A})$. Then there exist morphisms $\Psi \in \mathrm{Mor}(C, A \otimes C)$ and $\widehat{\Psi} \in \mathrm{Mor}(C, \widehat{A} \otimes C)$ satisfying the formulae appearing in conditions 2 and 3. In general they are not faithful. One can show that $\ker \Psi = \ker \widehat{\Psi}$ and that the quotient algebra

$$C/\ker \Psi = A \boxtimes A.$$

Theorem 43.1 Let $X \in \mathrm{C}^*_G$ and $C = X \boxtimes \widehat{A}$. Then there exists a faithful $\widehat{\Psi} \in \mathrm{Mor}(C, \widehat{A} \otimes C)$ such that

$$\widehat{\Psi}(\alpha_{X\widehat{A}}(x)) = I \otimes \alpha_{X\widehat{A}}(x)$$
$$\widehat{\Psi}(\beta_{X\widehat{A}}(\widehat{a})) = (\mathrm{id} \otimes \beta_{X\widehat{A}})\widehat{\Delta}(\widehat{a})$$

Indeed take $\widehat{\Psi} = \mathrm{id} \boxtimes \widehat{\Delta}$.

$\widehat{\Psi}$ is a left action of $\widehat{G}$ on $X \boxtimes \widehat{A}$. This is called dual action.

Theorem 43.1 *Landstad theory $(C, \widehat{\Psi}, \beta_{X\widehat{A}})$ is an example of G-product. In general*

Definition 43.2 G-product is a triple (C, β, ψ), consisting of a C^*-algebra C, a left continuous action $\beta \in \mathrm{Mor}(C, \widehat{A} \otimes C)$ of $\widehat{G}$ on C and an injective morphism $\psi \in \mathrm{Mor}(\widehat{A}, C)$ such that the diagram

$$\begin{tikzcd}
\widehat{A} \arrow[r, "\psi"] \arrow[d, "\widehat{\Delta}"'] & C \arrow[d, "\beta"] \\
\widehat{A} \otimes \widehat{A} \arrow[r, "\mathrm{id}\otimes\psi"'] & \widehat{A} \otimes C
\end{tikzcd}$$

is commutative.

Is any G-product of the form $(X \boxtimes \widehat{A}, \mathrm{id} \boxtimes \widehat{\Delta}, \beta_{X\widehat{A}})$? where $X \in \mathrm{C}^*_G$. What is the position if X inside $\mathrm{M}(X \boxtimes \widehat{A})$?
Satisfactory answer for regular quantum groups (Vaes). Landstad had it for classical locally compact groups. What about non-regular groups?
Let $R \in M(\widehat{A} \otimes \widehat{A})$ be a bicharacter:

$$(\mathrm{id} \otimes \widehat{\Delta})R = R_{12}R_{13},$$
$$(\widehat{\Delta} \otimes \mathrm{id})R = R_{23}R_{13}.$$

Given R one may consider functor $\boxtimes_R : \mathrm{C}^*_G \times \mathrm{C}^*_G \longrightarrow \mathrm{C}^*$.
Let $\alpha, \beta \in \mathrm{Rep}(A, H)$. We say that (α, β) is an R-Heisenberg pair if

$$(\star) \qquad V_{1\alpha}V_{2\beta} = V_{2\beta}V_{1\alpha}R_{12}.$$

- Does $\boxtimes_R : \mathrm{C}^*_G \times \mathrm{C}^*_G \longrightarrow \mathrm{C}^*_G$?
- Is $\boxtimes$ associative?
- Is C^*_G a monoidal category?
  Yes if R is an R-matrix. Any monoidal structure on C^*_G comes from an R-matrix.

Definition 43.3 We say that $R \in \mathrm{M}(\widehat{A} \otimes \widehat{A})$ is an R-matrix if

$$R_{12}V_{13}V_{23} = V_{23}V_{13}R_{12}$$

A locally compact quantum group $G = (A, \Delta)$ is called quasitriangular if there exists a unitary R-matrix in $M(\widehat{A} \otimes \widehat{A})$.